Kontsevich Invariants

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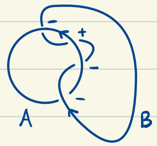
Kontsevich Integral

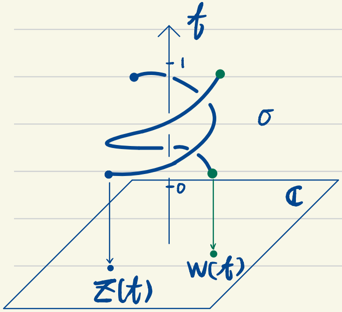
§1. From the simplest example : braiding

Question 1 : How to calculate link numbers ?



$lk(A,B) := \frac{1}{2}(\#\{+\} - \#\{-\}) = 1.$

Step I. The braiding number of a 2-braid.

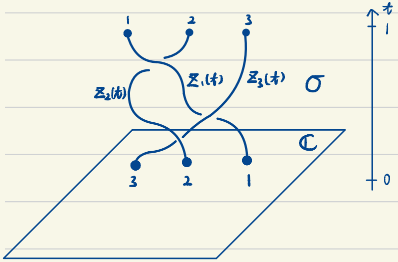


$$[\sigma] \in \pi_1(\mathrm{Conf}_2(\mathbb{C})/\Sigma_2)$$

$$\#\text{ of full twists in } \sigma = \frac{1}{2\pi\sqrt{-1}}\int_0^1 \frac{dz-dw}{z-w} \in \frac{1}{2}\mathbb{Z}$$

$$= \frac{1}{2\pi\sqrt{-1}}\int_0^1 d\log(z-w)$$

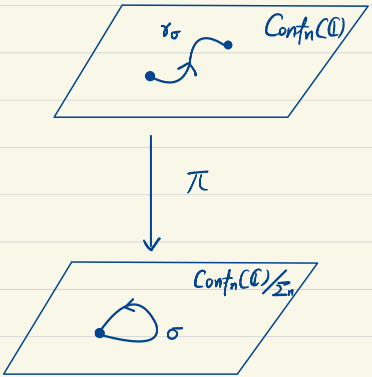
Step I'. The braiding numbers of an n-braid.



$[\sigma] \in \pi_1(\mathrm{Conf}_n(\mathbb{C})/\Sigma_n)$,

the braiding number of the i-th and j-th strands

$= \frac{1}{2\pi\sqrt{-1}} \int_0^1 d \log(z_i(t) - z_j(t))$



⇝ Denote formal variables $\{t_{ij} \mid 1 \leq i < j \leq n\}$,

$$\frac{1}{2\pi\sqrt{-1}} \int_0^1 \sum_{1 \leq i < j \leq n} t_{ij} \, d \log(z_i(t) - z_j(t)) \qquad (*)$$

⇝ Write $\Omega_n := \frac{1}{2\pi\sqrt{-1}} \sum_{1 \leq i < j \leq n} t_{ij} \, d \log(z_i - z_j)$

$$\in \Omega^1(\mathrm{Conf}_n(\mathbb{C}), \bigoplus_{1 \leq i < j \leq n} \mathbb{C} t_{ij})$$

$$(*) = \int_{\gamma_\sigma} \Omega_n = \int_0^1 \gamma_\sigma^* (\Omega_n)$$

Question 2 : Why is $\int_{\gamma_0} \Omega_n$ an invariant w.r.t horizontal deformations of $\mathbb{C}\times\mathbb{R}_t$?

Answer : $\boxed{d\Omega_n = 0}$.

If we write $Z_\gamma(t) := \exp\left(\int_0^t \gamma^*(\Omega_n)\right) \in \hat{T}\left(\bigoplus_{1\le i<j\le n} \mathbb{C}\, t_{ij}\right)$

$$\Longleftrightarrow \begin{cases} \frac{\partial}{\partial t} Z_\gamma(t) = \iota_{\dot\gamma}(\Omega_n)\, Z_\gamma(t) \qquad (**) \qquad \text{(KZ equation)} \\ Z_\gamma(0) = 1 \end{cases}$$

(holonomy w.r.t. ∇_n along γ)

Question 2' : When is $Z_\gamma(1)$ an invariant w.r.t horizontal deformations of $\mathbb{C}\times\mathbb{R}_t$?

$\Longleftrightarrow$ By which requirement on Ω_n, $Z_\gamma(1) = 0$ if γ is a contractible loop in $\mathrm{Conf}_n(\mathbb{C})$?

$\Longleftrightarrow$ When does the following equation

$$\begin{cases} (d - \Omega_n)\, W = 0, \qquad \text{(KZ equation)} \\ W(x_0) = 1 . \end{cases}$$

has a unique local solution valued in $\hat{T}\left(\bigoplus_{1\le i<j\le n} \mathbb{C}\, t_{ij}\right)$ near x_0 $\Longleftrightarrow$ integrable ?

(KZ connection)

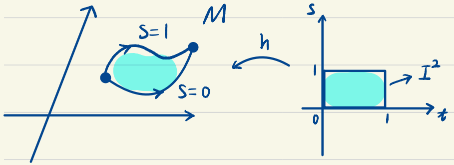
Write $\nabla_n := d - \Omega_n \wedge$ as a connection on $\mathrm{Conf}_n(\mathbb{C})$ valued in $\bigoplus_{1\le i<j\le n} \mathbb{C}\, t_{ij}$.

Answer: ∇_n is a flat connection, $\nabla_n^2 = 0$, i.e.

$$\boxed{d\,\Omega_n = \Omega_n \wedge \Omega_n .}$$

Proof for flatness $\Rightarrow$ integrability:

$h: I^2 \to M$, smooth, $h(0,s)$ and $h(1,s)$ constant $\Rightarrow p(0,s) = p(1,s) = 0$



Write $h^*\Omega_n = p\, ds + q\, dt$. Then

$d\Omega_n = \Omega_n \wedge \Omega_n \Rightarrow \frac{\partial p}{\partial t} - \frac{\partial q}{\partial s} = [q, p]$.

Denote $Z(t,s)$ be the unique solution of

$$\begin{cases} \frac{\partial Z}{\partial t} = q Z , \\ Z(0,s) = 1, \quad \forall s . \end{cases}$$

Then,

$$\frac{\partial^2 Z}{\partial t \partial s} = \frac{\partial}{\partial s}(qZ) = \frac{\partial p}{\partial t} Z - [q, p] Z + q \frac{\partial}{\partial s} Z$$

$$\Leftrightarrow \quad \frac{\partial}{\partial t}\left(\frac{\partial Z}{\partial s} - pZ\right) = q\left(\frac{\partial Z}{\partial s} - pZ\right)$$

$$\Leftrightarrow \quad \frac{\partial Z}{\partial s} - pZ = Z\left(0 - p(0,s)\right) = 0$$

$$\Leftrightarrow \quad \frac{\partial Z}{\partial s}(t,s) = p(t,s) Z(t,s) .$$

When $t = 1$, $\frac{\partial Z}{\partial s}(1,s) = 0$. □

Lem 1. $\Omega_n \wedge \Omega_n = 0$ if

- (2T relations)
  $[t_{ij}, t_{k\ell}] = 0, \quad \forall \#\{i,j,k,\ell\}=4, \quad i<j, \ k<\ell.$

- (4T relations)
  $[t_{ij}, t_{ik}] = -[t_{ij}, t_{jk}] = [t_{ik}, t_{jk}], \quad \forall \ i<j<k$

Denote $t_{ijk} = [t_{ij}, t_{jk}]$

pf. Write $\omega_{ij} := \frac{1}{2\pi\sqrt{-1}} \, d \log(z_i - z_j)$. Then

$$\begin{aligned}
\Omega_n \wedge \Omega_n &= \sum_{i<j,\, k<\ell} t_{ij} t_{k\ell} \, \omega_{ij} \wedge \omega_{k\ell} \\
&= \sum_{i<j<k} t_{ijk} \, (\omega_{ij} \wedge \omega_{jk} - \omega_{ik} \wedge \omega_{jk} - \omega_{ij} \wedge \omega_{ik}) \\
&\quad + \sum_{\#\{i,j,k,\ell\}=4} [t_{ij}, t_{k\ell}] \, \omega_{ij} \wedge \omega_{k\ell} \\
&= 0.
\end{aligned}$$

$\square$

(Arnold's identity) $\omega_{ij} \wedge \omega_{jk} - \omega_{ij} \wedge \omega_{ik} - \omega_{ik} \wedge \omega_{jk} = 0.$

pf: $\iff (dz_i \wedge dz_j + dz_j \wedge dz_k + dz_k \wedge dz_i) \cdot (z_k - z_i + z_j - z_k + z_i - z_j) = 0.$

$\square$

Ref. D. Bar-Natan,
"On the Vassiliev Knot Invariants."

Question 3 : How to write down $Z_\gamma(1) = \exp\left(\int_\gamma \Omega_n\right)$?

Ref: Francis Brown,
" Iterated integrals in QFT. "

Answer : $\exp\left(\int_0^t A(t)\,dt\right)$ (Dyson series ; iterated integral)

$$:= 1 + \sum_{N=1}^{\infty} \int_{0\leq t_1 \leq \cdots \leq t_N \leq t} A(t_N)\,A(t_{N-1}) \cdots A(t_1)\,dt_1 \cdots dt_N$$

is the unique solution to the following equation

$$\begin{cases} \frac{d}{dt} Z(t) = A(t)\,Z(t), \\ Z(0) = 1. \end{cases}$$

Proof for Dyson series :

- Dyson series converges absolutely for $t \in [0,T]$,

$$\left|\int_{0\leq t_1 \leq \cdots \leq t_N \leq t} A(t_N) \cdots A(t_1)\,dt_1 \cdots dt_N\right| \leq \sup_{0\leq s \leq T} |A(s)|^n \frac{t^n}{n!}.$$

- $\frac{d}{dt}\left(1 + \sum_N \int_{0\leq t_1 \leq \cdots \leq t_N \leq t} A(t_N) \cdots A(t_1)\,dt_1 \cdots dt_N\right)$

$$= \sum_N A(t) \int_{0\leq t_1 \leq \cdots \leq t_{N-1} \leq t} A(t_{N-1}) \cdots A(t_1)\,dt_1 \cdots dt_{N-1}$$

$$= A(t)\,Z(t).$$

- Uniqueness is by Picard's method using successive approxiamation.

$\square$

Hence,

$$\boxed{\begin{aligned} Z_\gamma(1) = 1 + \sum_{N=1}^{\infty} \left(\frac{1}{2\pi\sqrt{-1}}\right)^N \int_{0 \leq t_1 \leq \cdots \leq t_N \leq 1} & \sum_{\{(i_k, j_k),\, k=1,\cdots,N \,|\, 1 \leq i_k < j_k \leq n\}} \\ & t_{i_N, j_N} \cdots t_{i_1, j_1} \frac{dz_{i_1} - dz_{j_1}}{z_{i_1} - z_{j_1}} \wedge \cdots \wedge \frac{dz_{i_N} - dz_{j_N}}{z_{i_N} - z_{j_N}} \quad (***) \end{aligned}}$$

Def 1. The Lie algebra (Drinfel'd - Kohno Lie algebra) $\mathfrak{t}_n$ is defined as,

$$\mathfrak{t}_n := \mathrm{FreeLie}_{\mathbb{C}} \langle t_{ij} \mid 1 \leq i < j \leq n \rangle \, / \, (2T, 4T).$$

We shall regard $Z_\gamma(1)$ valued $\widehat{U(\mathfrak{t}_n)}$.

Example. $\mathfrak{t}_2 = \mathbb{C} t_{12} \qquad \widehat{U(\mathfrak{t}_2)} = \mathbb{C}[\![t_{12}]\!]$.

↙ central element.

$$\mathfrak{t}_3 = \mathrm{FreeLie}_{\mathbb{C}} \langle t_{12}, t_{23} \rangle \oplus \mathbb{C}(t_{12} + t_{13} + t_{23})$$

$$\widehat{U(\mathfrak{t}_3)} = \mathbb{C}[\![t_{12} + t_{13} + t_{23}]\!] \langle\!\langle t_{12}, t_{23} \rangle\!\rangle.$$

→ free algebra

Degree completion

$\mathfrak{t}_n$ is graded Lie algebra with

$$\deg(t_{ij}) = 1.$$

Notice that,

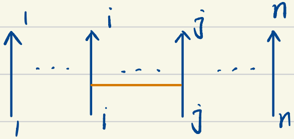
- The completions $\widehat{U(\mathfrak{t}_n)}$ and $\hat{\mathfrak{t}}_n$ are all w.r.t. this degree.
- $\widehat{U(\mathfrak{t}_n)} = \hat{U}(\hat{\mathfrak{t}}_n)$.

Remark

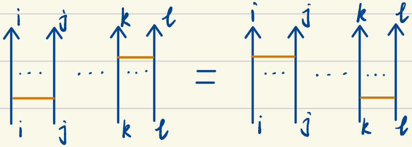
Magic : $\widehat{U(\mathfrak{t}_n)} \cong \hat{\mathcal{A}}^h(\overset{1}{\underset{1}{\uparrow}}\overset{2}{\underset{2}{\uparrow}} \cdots \overset{n}{\underset{n}{\uparrow}}) \hookrightarrow \hat{\mathcal{A}}(\overset{1}{\underset{1}{\uparrow}}\overset{2}{\underset{2}{\uparrow}} \cdots \overset{n}{\underset{n}{\uparrow}}).$

(Horizontal chord diagrams)

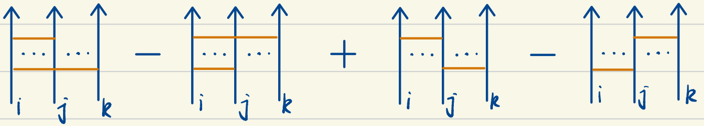
$t_{ij} \sim$



(2T-relations)



(4T-relations)

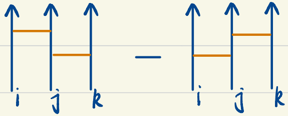


$= 0.$

Hence, the holonomy $Z_\gamma(1)$ is valued in $\hat{\mathcal{A}}^h(\overset{1}{\underset{1}{\uparrow}}\overset{2}{\underset{2}{\uparrow}} \cdots \overset{n}{\underset{n}{\uparrow}})$.

$t_{ijk} := [t_{ij}, t_{jk}]$

$\sim$



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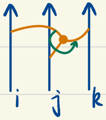


In summary,

$$\Omega_n := \sum_{1\le i<j\le n} t_{ij}\, \omega_{ij} \in \Omega^1(\mathrm{Conf}_n(\mathbb{C}), \mathfrak{t}_n) \text{ satisfies}$$

- $d\Omega_n = 0$,
- $\Omega_n \wedge \Omega_n = 0$.

$\Rightarrow$ $\nabla_n := d - \Omega_n$ is a flat connection.

$\Rightarrow$ Given initial data, $\nabla_n W = 0$ defines a multivalued flat section

$$W : \mathrm{Conf}_n(\mathbb{C}) \longrightarrow \hat{\mathcal{A}}^h(\uparrow_1 \cdots \uparrow_n),$$

whose holonomy defines a functor between groupoids,

$$Z : \Pi_1(\mathrm{Conf}_n(\mathbb{C})) \longrightarrow \mathcal{G}_n$$
$$\underline{z} \xrightarrow{\gamma} \underline{w} \longmapsto \underline{z} \xrightarrow{Z_\gamma(1)} \underline{w},$$

where $\mathcal{G}_n$ is a groupoid with

- $\mathrm{Obj}(\mathcal{G}_n) = \mathrm{Conf}_n(\mathbb{C})$ as a set.
- $\mathcal{G}_n(\underline{z}, \underline{w}) = \{$ Group-like elements in $\widehat{U(\mathfrak{t}_n)}\}$ $= \exp(\hat{\mathfrak{t}}_n)$.

Remark. Especially, we have group homomorphisms.

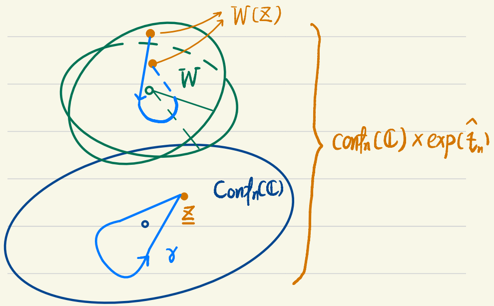
$$Z : \quad PB_n \longrightarrow \exp(\hat{\mathfrak{t}}_n)$$
$$B_n \longrightarrow \exp(\hat{\mathfrak{t}}_n) \rtimes \Sigma_n$$

Def-Lem: For a 'nice' Hopf algebra H, $g \in H$ is called group-like if

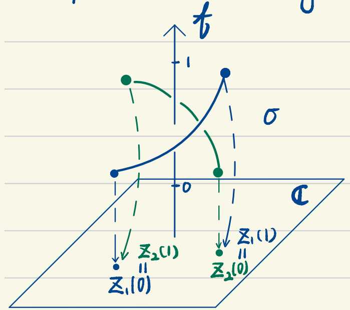
- $\Delta g = g \hat{\otimes} g$
- (Equivalently) $g = \exp(X)$, for some $X \in \mathrm{Prim}(H)$.

The set of group-like elements is a group.
(Baker - Cambell - Hausdorff) ↗

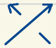
$$\exp(X)\exp(Y) = \exp(BCH(X,Y)).$$



Example. Let's go back 2-braids!



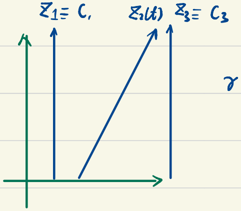
σ is a path $(Z_1(t), Z_2(t))$ for $0 \leq t \leq 1$.

$$\begin{cases} Z_1(0) = Z_2(1), \\ Z_2(0) = Z_1(1), \\ [\sigma] = \end{cases}$$

$\in B_2$.

Then, $Z : \left((Z_1(0), Z_2(0)) \xrightarrow{\sigma} (Z_1(1), Z_2(1)) \right) \longmapsto \left((Z_1(0), Z_2(0)) \xrightarrow{Z_\sigma(1)} (Z_1(1), Z_2(1)) \right)$

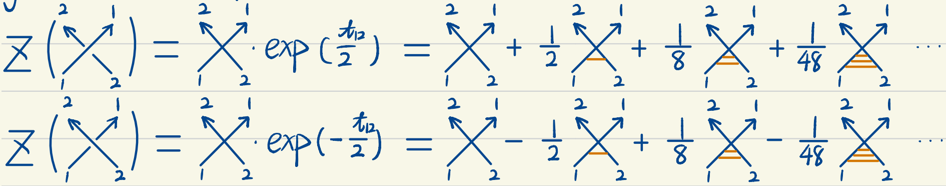
$$Z_\sigma(1) = 1 + \sum_{N=1}^{\infty} \left(\frac{1}{2\pi\sqrt{-1}} \right)^N \int_{0 \leq t_1 \leq \cdots \leq t_N \leq 1} t_{12}^N \bigwedge_{i=1}^{N} \frac{dz_1(t_i) - dz_2(t_i)}{z_1(t_i) - z_2(t_i)}$$

$$= 1 + \sum_{N=1}^{\infty} \frac{1}{N!\, 2^N} t_{12}^N = \exp\left(\frac{t_{12}}{2} \right) = \mathcal{R} \in \hat{\mathcal{A}}^h(\uparrow\uparrow)$$



$Z(\gamma) \neq 0$!

However, if you want to include the information of the objects, you need to write it as



§2. Towards algebraic structures: Compactification

Question 4: How to make an algebraic theory from this integral?

Slogan: Algebras come from compactifications.

- Adding boundary points to describe how points collide.
- Boundary points give algebraic information.
- Boundary points with highest codimension are purely algebraic.

Step II. • Extending the holonomy functor Z onto the compactified configuration space,

• and restricting it between boundary points with highest codimension.

Remark. Notice that Z is invariant w.r.t. translations and dilation over $\mathbb{C}$, which means that we can replace $\mathrm{Conf}_n(\mathbb{C})$ by,

$$C_n(\mathbb{C}) := \mathrm{Conf}_n(\mathbb{C}) / \mathbb{C} \rtimes \mathbb{R}_{>0}.$$

$$C_1(\mathbb{C}) = \{*\}, \quad C_2(\mathbb{C}) = S^1.$$

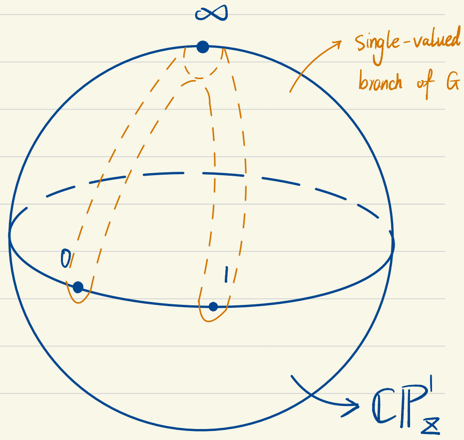
Ref: B. Fresse,
"Homotopy of operads and Grothendieck - Teichmüller Groups".

The first non-trivial case is $n=3$ case.

Write

$$\overset{\text{central element.}}{c} = \frac{t_{12}+t_{23}+t_{13}}{2\pi\sqrt{-1}}, \quad x = \frac{t_{12}}{2\pi\sqrt{-1}}, \quad y = \frac{t_{23}}{2\pi\sqrt{-1}},$$

Then $\widehat{U(t_3)} = \mathbb{C}[c]\langle\langle x, y\rangle\rangle$. We calculate the multivalued solution of KZ equation on $C_3(\mathbb{C})$.



$$0=(d-\Omega_3)W = dW - \frac{1}{2\pi\sqrt{-1}}\Big(t_{12}\, d\log\frac{z_2-z_1}{z_3-z_1} + t_{23}\, d\log\frac{z_3-z_2}{z_3-z_1} + c\, d\log(z_3-z_1)$$

Denote $z := \frac{z_2-z_1}{z_3-z_1}$

$$\Leftrightarrow \quad 0 = dW - (x\, d\log z + y\, d\log(1-z) + c\, d\log(z_3-z_1))W$$

$$\Leftrightarrow \quad ((z_3-z_1)^{-c}W)' = \left(\frac{x}{z} + \frac{y}{z-1}\right)((z_3-z_1)^{-c}W)$$

Denote $W = (z_3-z_1)^c G(z) = \sum_{k\geq 0} \frac{(\log(z_3-z_1))^k}{k!} c^k G(z)$,

$$\Leftrightarrow \quad G'(z) = \left(\frac{x}{z} + \frac{y}{z-1}\right) G(z) \qquad (*)$$

(*) has three regular singularities at $z = 0, 1, \infty$.

$$\partial\overline{C_3(\mathbb{C})} = \{z_1=z_2\} \sqcup \{z_2=z_3\} \sqcup \{z_1=z_3\}$$

Take

$$U = \mathbb{CP}^1 \setminus [0,\infty] \cup [1,\infty].$$

U is a single-valued branch for $G(z)$ valued in $\mathbb{C}\langle\langle x, y\rangle\rangle$.

$\Rightarrow$ Any two non-zero solutions on U are differed by a constant element in $\mathbb{C}\langle\langle x, y\rangle\rangle$.

- Near $z=0$, $(*)$ is in the form

$$z\frac{d}{dz}G=\Big(x-\sum_{k\geq 1}yz^k\Big)G \qquad (*_0)$$

<u>Claim</u>: We have a solution near $z=0$ in the form

$$\boxed{G_0(z)=\Big(1+\sum_{k\geq 1}g_kz^k\Big)z^x,}$$

where $g_k\in\mathbb{C}\langle\langle x,y\rangle\rangle$.

pf. $(*_0)\iff \sum_{k\geq 1}kg_kz^k+\Big[1+\sum_{k\geq 1}g_kz^k,x\Big]+\sum_{l\geq 1}yz^l\Big(1+\sum_{k\geq 1}g_kz^k\Big)=0$

$\iff kg_k-\mathrm{ad}_x g_k=-\sum_{l=1}^{k-1}yg_l-y$

$\iff g_k=(\mathrm{ad}_x-k)^{-1}y\Big(1+\sum_{l=1}^{k-1}g_l\Big).$

ad_x-k is invertible in $\mathbb{C}\langle\langle x,y\rangle\rangle$, i.e.,

$$(\mathrm{ad}_x-k)^{-1}=-\sum_{l\geq 0}\frac{\mathrm{ad}_x^{l}}{k^{l+1}}$$

By this recursion relation, we can get g_k, $\forall k\geq 1$. □

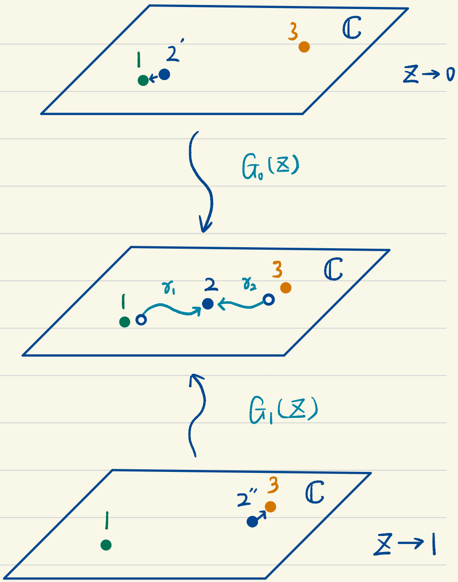
- Near $z=1$, $(*)$ is in the form $\nearrow$ $z':=1-z$

$$z'\frac{d}{dz'}G=\Big(y-\sum_{k\geq 1}xz'^k\Big)G \qquad (*_1)$$

<u>Claim</u>: We have a solution near $z=1$ in the form

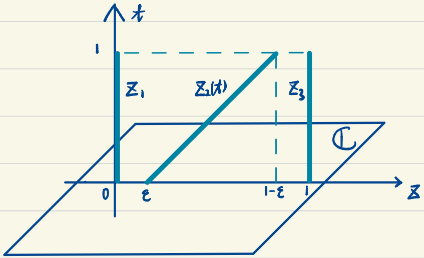
$$G_1(z)=\Big(1+\sum_{k\geq 1}h_k(1-z)^k\Big)(1-z)^y$$

where $h_k\in\mathbb{C}\langle\langle x,y\rangle\rangle$.



Def 2. The KZ associator is an element in $\mathbb{C}\langle\langle t_{12}, t_{23}\rangle\rangle$ defined

$$\Phi_{KZ} := G_1^{-1}(z) \cdot G_0(z).$$



$\varepsilon \in \mathbb{R} \cap (0,1)$

Explanation. Consider a path γ_ε in $C_3(\mathbb{C})$ moving from $(0, \varepsilon, 1)$ to $(0, 1-\varepsilon, 1)$ with z_1 and z_3 fixed. Recall that for any locally-defined solution W of KZ equation

defined by Kontsevich integral.

$$W(0, 1-\varepsilon, 1) = Z(\gamma_\varepsilon) W(0, \varepsilon, 1),$$

Hence

$$\begin{aligned}\Phi_{KZ} &= G_1^{-1}(1-\varepsilon)\ Z(\gamma_\varepsilon)\ G_0(\varepsilon) \\ &= \varepsilon^{-y}\ \Big(1 + \sum_{k\geq1} h_k \varepsilon^k\Big)^{-1} Z(\gamma_\varepsilon)\ \Big(1 + \sum_{k\geq1} g_k \varepsilon^k\Big)\ \varepsilon^{x} \\ &= \lim_{\varepsilon\to0_+} \varepsilon^{-\frac{t_{23}}{2\pi\sqrt{-1}}}\ Z(\gamma_\varepsilon)\ \varepsilon^{\frac{t_{12}}{2\pi\sqrt{-1}}}. \qquad (**)\end{aligned}$$

$\Longrightarrow$ $\Phi_{KZ}(t_{12}, t_{23})$ are extension of Z between boundary points of $\overline{C_3(\mathbb{C})}$.

However, by similar construction, we can write down $(**)$ on $\overline{C_n(\mathbb{C})}$, $\forall n \geq 3$.

Basic properties of Φ_{KZ}.

- $\Phi_{KZ}(t_{12}, t_{23})$ is a group-like (of course invertible) element , i.e.,

$$\Phi_{KZ}(t_{12}, t_{23}) \in \exp\big(\widehat{\text{Free Lie}\langle t_{12}, t_{23}\rangle}\big) \hookrightarrow \exp(\hat{t}_3).$$

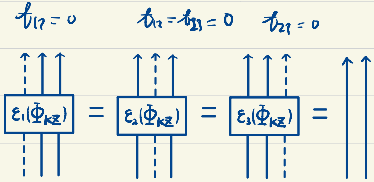
pf : $\Phi_{KZ} = G_1^{-1} G_0$ and G_1, G_0 are group-like. □

- $\Phi_{KZ} \in \hat{U}(\hat{t}_3) \xmapsto{\text{PBW}} 1 \in \hat{S}(\hat{t}_3)$. only constant term survives.

pf : If $[x, y] = 0$, $G_1(z, x, y) = G_0(z, x, y) = z^x (1-z)^y$. □

What's more, $\Phi_{KZ}(0, t_{23}) = \Phi_{KZ}(t_{12}, 0) = \Phi_{KZ}(0, 0) = 1$.

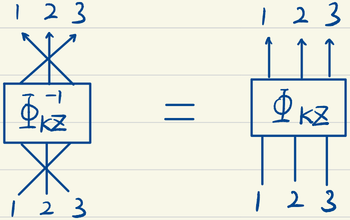
$$\boxed{\varepsilon_1 \Phi_{KZ} = \varepsilon_2 \Phi_{KZ} = \varepsilon_3 \Phi_{KZ} = 1} \in \hat{U}(\hat{t}_2)$$



- Notice that

$$G_0(z, x, y) = G_1(1-z, y, x)$$

$$\begin{aligned}\Phi_{KZ}(t_{12}, t_{23}) &= G_1(z, x, y)^{-1} G_0(z, x, y) \\ &= G_0(1-z, y, x)^{-1} G_1(1-z, y, x) \\ &= \Phi_{KZ}^{-1}(t_{23}, t_{12}).\end{aligned}$$



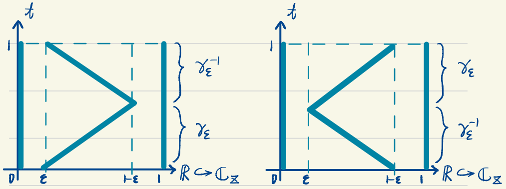
Another proof :

$$\Phi_{KZ}(t_{23}, t_{12})\, \Phi_{KZ}(t_{12}, t_{23}) = \lim_{\varepsilon\to 0+} \varepsilon^{-x} Z_{\gamma_\varepsilon^{-1}} \varepsilon^{y} \varepsilon^{-y} Z_{\gamma_\varepsilon} \varepsilon^{x} = 1.$$

$\nearrow \Phi_{KZ}^{321}$ $\quad \nearrow \Phi_{KZ}^{123}$

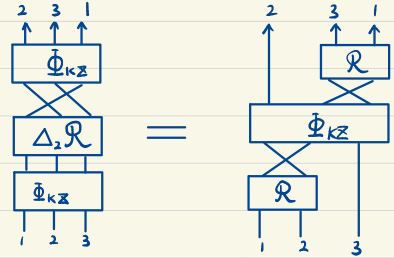
$$\Phi_{KZ}(t_{12}, t_{23})\, \Phi_{KZ}(t_{23}, t_{12}) = \lim_{\varepsilon\to 0+} \varepsilon^{-y} Z_{\gamma_\varepsilon} \varepsilon^{x} \varepsilon^{-x} Z_{\gamma_\varepsilon^{-1}} \varepsilon^{y} = 1.$$

$$\boxed{\Phi_{KZ}^{-1} = \Phi_{KZ}^{321}.}$$



- Hexagon relations

$$\boxed{\begin{cases} \Phi_{KZ}^{231}\,(\Delta_2 \mathcal{R})\,\Phi_{KZ} = \mathcal{R}^{13}\,\Phi_{KZ}\,\mathcal{R}^{12} & \\ (\Phi_{KZ}^{-1})^{312}(\Delta_1 \mathcal{R})\,\Phi_{KZ}^{-1} = \mathcal{R}^{13}\,(\Phi_{KZ}^{-1})^{132}\mathcal{R}^{23} & \end{cases}} \qquad \begin{matrix}(H_R)\\(H_L)\end{matrix}$$



pf. Firstly, $(H_R) \xleftrightarrow{\tau_{13}} (H_L)$, we only need to prove (H_R).

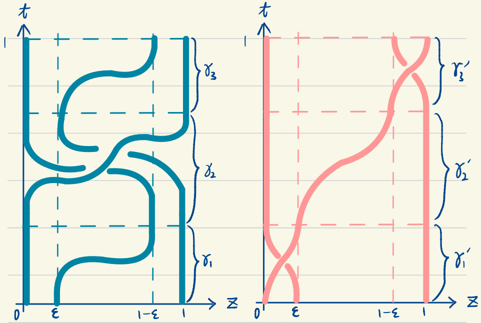
Notice that

$$Z(\gamma_3)Z(\gamma_2)Z(\gamma_1) = Z(\gamma_3')Z(\gamma_2')Z(\gamma_1'),$$

since $\gamma_1^{-1}\gamma_2^{-1}\gamma_3^{-1}\gamma_3'\gamma_2'\gamma_1'$ is a null-homotopic loop.

We only need to show that

$$\begin{cases} \lim_{\varepsilon\to 0_+} \varepsilon^{-\frac{t_{23}}{2\pi\sqrt{-1}}} Z(\gamma_2)\varepsilon^{\frac{t_{23}}{2\pi\sqrt{-1}}} = \Delta_2 \mathcal{R}, & (1) \\ \lim_{\varepsilon\to 0_+} \varepsilon^{-\frac{t_{12}}{2\pi\sqrt{-1}}} Z(\gamma_1')\varepsilon^{\frac{t_{12}}{2\pi\sqrt{-1}}} = \mathcal{R}^{12}, & (2) \\ \lim_{\varepsilon\to 0_+} \varepsilon^{-\frac{t_{13}}{2\pi\sqrt{-1}}} Z(\gamma_3')\varepsilon^{\frac{t_{13}}{2\pi\sqrt{-1}}} = \mathcal{R}^{13}. & (3) \end{cases}$$



They are proved by choosing local solutions.

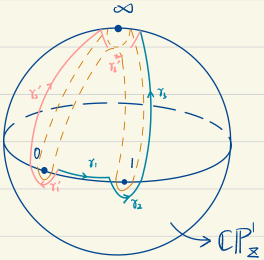
For (2), take $W(z_1,z_2,z_3) = \underbrace{g(z_1,z_2,z_3)}_{\rightarrow\text{ holomorphic, and } \sim (z_3-z_1)^c \text{ (as } z_2\to z_1)}\left(\frac{z_2-z_1}{z_3-z_1}\right)^{\frac{t_{12}}{2\pi\sqrt{-1}}}$

$$Z(\gamma_1') = g(\varepsilon,0,1)\left(\frac{\varepsilon}{1-\varepsilon}\right)^{\frac{t_{12}}{2\pi\sqrt{-1}}}\underbrace{\exp\left(\frac{t_{12}}{2}\right)}_{\rightarrow \mathcal{R}^{12}.}\varepsilon^{-\frac{t_{12}}{2\pi\sqrt{-1}}}g(0,\varepsilon,1)^{-1}$$

For (3), Similar as (2).

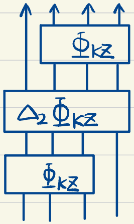
For (1), $Z(\gamma_2) = \underbrace{h(1,0,\varepsilon)}_{\text{holomorphic, and} \sim (z_3-z_1)^c \text{ (as } z_2\to z_3)}\left(\frac{\varepsilon}{1-\varepsilon}\right)^{\frac{t_{23}}{2\pi\sqrt{-1}}}\exp\left(-\frac{t_{23}}{2}\right)\varepsilon^{-\frac{t_{23}}{2\pi\sqrt{-1}}}h^{-1}(0,1-\varepsilon,1)$

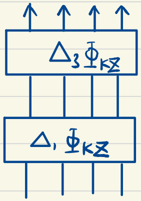
LHS of (1) $= \exp\left(\frac{t_{12}+t_{13}+t_{23}}{2}\right)\exp\left(-\frac{t_{23}}{2}\right) = \exp\left(\frac{t_{12}+t_{13}}{2}\right) = \Delta_2\mathcal{R}.$ □



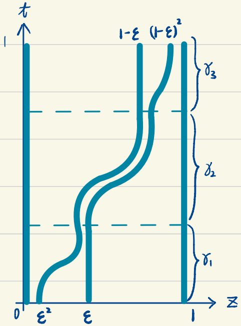
- Pentagon relation.

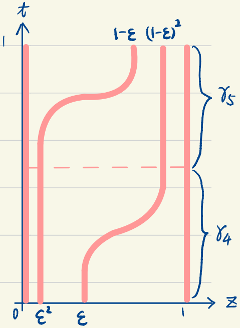
$$\boxed{\Phi_{KZ}^{234}\cdot\Delta_2\Phi_{KZ}\cdot\Phi_{KZ}^{123} = \Delta_3\Phi_{KZ}\cdot\Delta_1\Phi_{KZ}}$$





We leave the proof for Xinxing Tang since we need further information for KZ equation on $C_4(\mathbb{C})$ or Kontsevich integral of 4-strands.





Def 3. An associator Φ is a group-like invertible element in $\widehat{\mathcal{A}}(\uparrow_3)$ satisfies,

1). $\varepsilon_1\Phi = \varepsilon_2\Phi = \varepsilon_3\Phi = 1$,

2). $\Phi^{-1} = \Phi^{321}$,

3). $\Phi^{234}\cdot\Delta_2\Phi\cdot\Phi^{123} = \Delta_3\Phi\cdot\Delta_1\Phi$,

4). $\Phi^{231}(\Delta_2 R)\Phi = R^{13}\Phi R^{12}$.

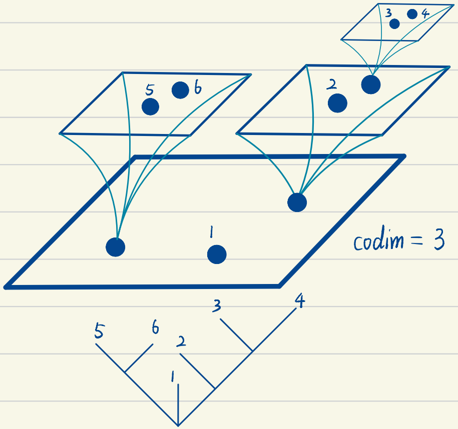
Thm 2. The KZ associator is an associator.

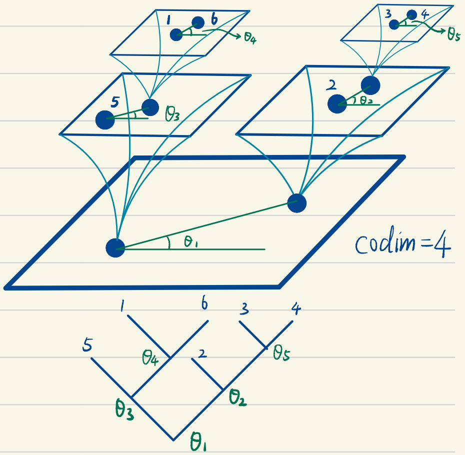
Algebraic version

If we take Fulton - MacPherson compactification $\overline{C_n(\mathbb{C})}$, there will be a decomposition property for $\partial\overline{C_n(\mathbb{C})}$, which means that,

$$\partial\overline{C_n(\mathbb{C})} = \bigcup_{n>k\geq 2, \{J_1,\cdots,J_k\}} \overline{C_k(\mathbb{C})} \times \prod_{i=1}^{k} \overline{C_{J_i}(\mathbb{C})},$$

where $\{J_1,\cdots,J_k\}$ is any partition of $\{1,\cdots,n\}$.





Boundary points with highest codimension is just a binary tree with labellings information on leaves and angles information on nodes.

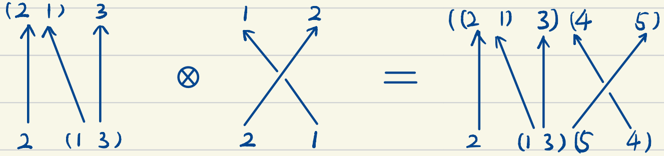
Require all the marked points staying on $\mathbb{R} \hookrightarrow \mathbb{C} \Leftrightarrow \theta_i = 0, \forall i.$

We get a subgroupoid PaB_n of $\pi_1(\overline{C_n(\mathbb{C})})$ with

- Obj. Parenthesized words of $\{1, \cdots, n\}$, Pa_n
- Mor. Parenthesized braids.

Def 4. The groupoid by putting all PaB_n together with an obvious monoidal structure is called the monoidal groupoid of parenthesized braids and is denoted PaB. This is a submonoidal category of q-tangles.

Example



$\varphi \in PaB_3$ $\gamma \in PaB_2$ $\varphi \otimes \gamma \in PaB_5$.

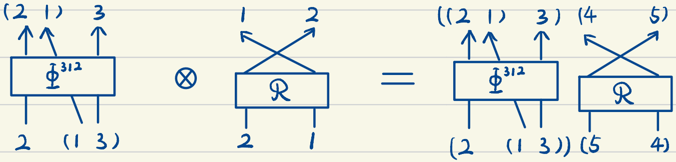
Notice that we can extend the holonomy functor Z onto PaB by the KZ associator. However, we also need to define the target groupoid of Z.

We get another groupoid $\mathcal{G}\mathrm{PaCD}_n$ by changing the morphism sets with group-like elements in $\widehat{\mathcal{U}(\mathfrak{t}_n)} \cong \widehat{\mathcal{A}^h(\uparrow_n)}$. i.e., in $\mathcal{G}\mathrm{PaCD}_n$

- Obj. Parenthesized words of $\{1, \cdots, n\}$, Pa_n
- Mor. Group-like elements in $\widehat{\mathcal{U}(\mathfrak{t}_n)}$.

Def 5. The groupoid by putting all $\mathcal{G}\mathrm{PaCD}_n$ together with an obvious monoidal structure is called the monoidal groupoid of parenthesized chord diagrams and is denoted by $\mathcal{G}\mathrm{PaCD}$.

Example



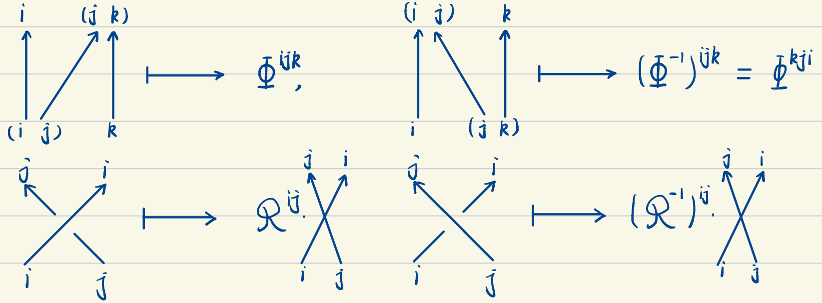
Then the holonomy functor defined a functor

$$Z : \mathrm{PaB} \longrightarrow \mathcal{G}\mathrm{PaCD},$$

which is called the Kontsevich integral on PaB.

This is a purely algebraic object, we can give it another definition.

Def 6. A (combinatorial) Kontsevich integral Z_{Φ} is a functor defined on PaB and valued in $\mathcal{G}$PaCD, identical on objects and generated by



and compatible with the doubling operator

$$Z_{\Phi} \circ \Delta_i = \Delta_i \circ Z_{\Phi} .$$

Thm3. $Z_{\Phi_{KZ}} = Z$.

pf. i) Z is monoidal.

$$Z(\gamma_1 \otimes_{\varepsilon} \gamma_2) = Z(\gamma_1) \otimes Z(\gamma_2) + O(\varepsilon).$$

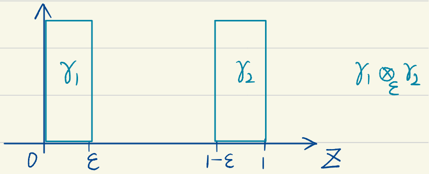
ii). Z is compatible with doubling.

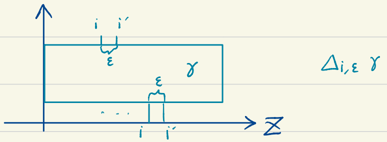
$$\lim_{\varepsilon \to 0} Z(\Delta_{i,\varepsilon} \gamma) = \Delta_i Z(\gamma) .$$

iii). General limit formula for ε-parameterized tangles.

Ref. S. Chmutov, S. Duzhin, J. Mostovoy, "Intro. to Vassiliev Knot Invariants". Thm 10.3.7. □

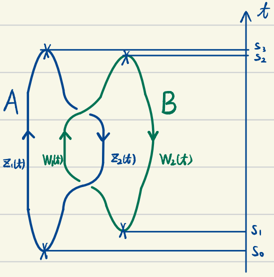
Rem. In fact, $\{\mathrm{PaB}_n\}$ and $\{\mathcal{G}\mathrm{PaCD}_n\}$ have an operad structure. Although PaB is not $\mathbb{C}$-linear, we can hope Z_{Φ} should be extended to an isomorphism between $\widehat{\mathrm{PaB}(k)}$ and $\mathcal{G}$PaCD as operads in $\mathbb{C}$-Groupoid.





§3. Return to links : Normalization

Question 1 : How to calculate link numbers ?



Step III
- Embedding a link 'nicely' into $\mathbb{C} \times \mathbb{R}_t$
- Splitting it into strands by cutting critical value
- Calculating Kontsevich integral for each pair of strand
- "±" for changing orientation and recording chords on wilson loops.

[Kontsevich] For 'nicely' embedded tangles T,

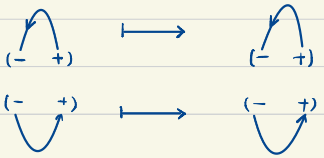
$$Z(T) := \sum_{m=0}^{\infty} \left(\frac{1}{2\pi\sqrt{-1}}\right)^m \int_{\substack{t_{min} < t_m < \cdots < t_1 < t_{max} \\ t_j \text{ are not critical}}} \sum_{P = \{(z_j, z_j')\}} (-1)^{\downarrow P} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'}$$

where we sum over all the set P of m pairs of strands

$\downarrow P$ = # of ↓ direction strand in P.

D_P are chord diagrams using the tangle T as the pattern and draw horizontal chords in the order of P and regarded as an element in $\widehat{\mathcal{A}^c(Patt(T))} / (1T)$.

Def 7. A (combinatorial) Kontsevich invariant Z_Φ on q-tangles$^{f.}$ T is extended by Z_Φ on PaB valued in $\widehat{\mathcal{A}^c(T)}$ satisfying,

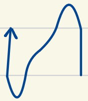


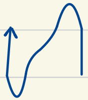
and

change orientation

$$Z_\Phi \circ S_i = S_i \circ Z_\Phi .$$

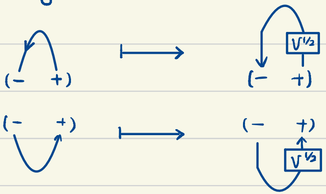
Unfortunately, Z_Φ is not a well-defined functor because

$Z_\Phi($  $) =$  $\neq \uparrow = Z_\Phi(\uparrow)$.

Let  $= \Big(Z_\Phi($  $)\Big)^{-1} \in \widehat{\mathcal{A}^c(\uparrow)}$

Notice that the leading term of  is 1, ν is well defined as a power series and there is a unique way to write done a square root of ν with leading term 1.

Def 7.' A normalized Kontsevich invariant $\check{Z}_\Phi$ on q-tangles$^{f.}$ T is extended by Z_Φ on PaB valued in $\widehat{\mathcal{A}}^c(T)$ satisfying

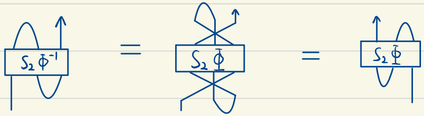


and

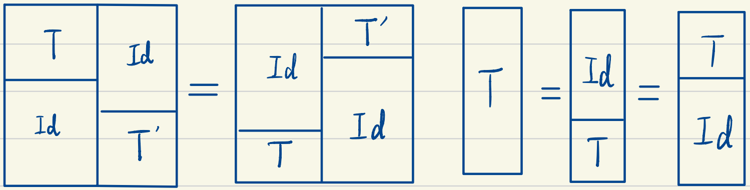
$$\check{Z}_\Phi \circ \overset{\text{change orientation}}{S_i} = S_i \circ \check{Z}_\Phi .$$

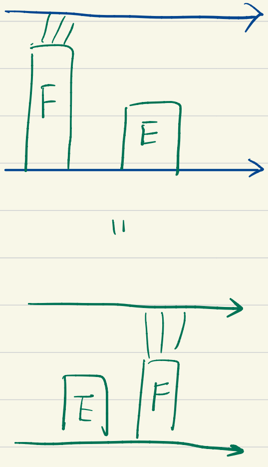
Thm 4. $\check{Z}_\Phi$ defined as above is an isotopy invariant of a framed oriented quasi-tangle T.

Pf. Since $\check{Z}_\Phi$ is extended from a strict monoidal functor on PaB, and compatible w.r.t. changing-orientation operations, we only need to show that $\check{Z}_\Phi$ is invariant under the following Turaev moves for framed oriented q-tangles.

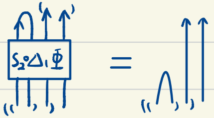


- (qFT_0)

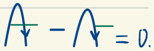




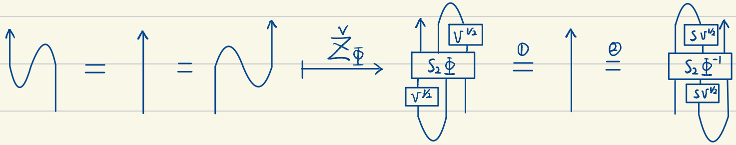
Almost all identities can be shown trivially, except for



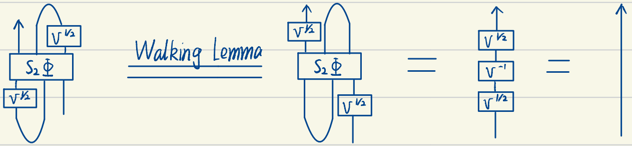
, This is by the sign convention,



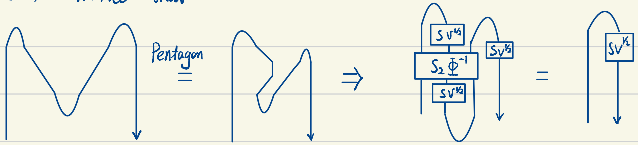
- (qFT_1)



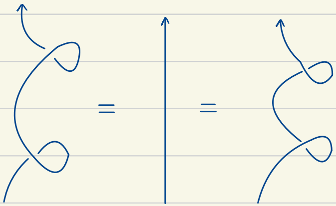
For ①,



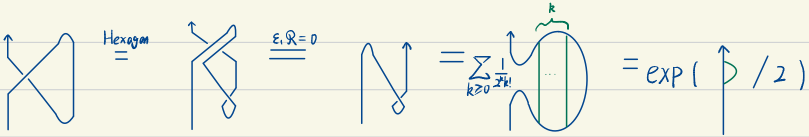
For ②, notice that



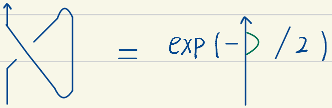
- (qFT$_4$)



Notice that, under $\check{Z}_\Phi$

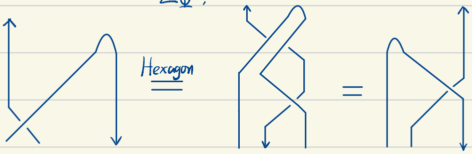


and dually,



(qFT$_5$)

For ③, under $\check{Z}_\Phi$,



For ④, similar.

□

(Hard!)

Thm 5 $\check{Z}_{\Phi}$: { framed on n-component links} $\longrightarrow \widehat{\mathcal{A}^c(\circlearrowleft_n)}$

is independent of Φ.

[Le - Murakami - Ohtsuki]

Thm 6. $\check{Z}(\overset{\checkmark\,\text{link}}{L})$ is group-like, i.e., as an element in $\widehat{\mathcal{A}^c(\circlearrowleft_n)}$

$$\Delta \check{Z}(L) = \check{Z}(L) \,\hat{\otimes}\, \check{Z}(L).$$

Coro. Since $\widehat{\mathcal{A}(\circlearrowleft)}$ is a completed graded Hopf algebra,

$\nu^{-1}\check{Z}(K) = \exp(z_K)$, with $z_K \in \mathrm{Prim}(\widehat{\mathcal{A}(\circlearrowleft)})$, which is a summation of Jacobi diagrams with connected unitrivalent graphs.

One can read knot invariants from coefficients in z_K.