Reshetikhin-Turaev Invariant

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8/24

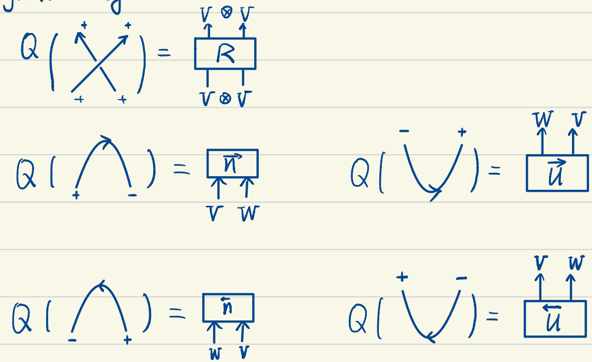
<u>Recall.</u>

- We have defined the rigid strict monoidal category $(\mathcal{T}, \otimes, \phi)$ of equivalent classes of (oriented) tangles.

- Operator invariants of tangles are regarded as strict monoidal functors.

$$Q = Q_{V,W,R,\vec{u},\vec{n},\overleftarrow{u},\overleftarrow{n}} : \mathcal{T} \longrightarrow \mathrm{Vect}_{\mathbb{k}}^{f}.$$

generated by



- By rigidity of $\mathcal{T}$ and $\mathrm{Vect}_{\mathbb{k}}^{f}$, $\vec{u}, \vec{n}, \overleftarrow{u}, \overleftarrow{n}$ are determined by a pair of isomorphisms.

$$\alpha, \beta : W^{*} \xrightarrow{\simeq} V.$$

and their difference $\mu := \beta \circ \alpha^{-1} : V \to V$, satisfies:

a). $R^{\pm} \circ (\mu \otimes \mu) = (\mu \otimes \mu) \circ R^{\pm}$

b). $\mathrm{Tr}_2 (R^{\pm} \circ (id_V \otimes \mu)) = id_V$

c). $(\tau \circ R^{-1})^{t_1} \circ (id_{V^*} \otimes \mu) \circ (R \circ \tau)^{t_1} \circ (id_{V^*} \otimes \mu^{-1}) = id_{V^*} \otimes id_V.$

- Invariants of links can be recovered from operator invariants of tangles via

$$Q_{V,V^*,R,\alpha,\beta}(L)(1_{\mathbb{k}}) = P_{V,R,\mu}(L).$$

Rem:

i). In fact, we can't define $Q : \mathcal{T} \longrightarrow \mathrm{Vect}_{\mathbb{K}}^{\dagger}$ directly, because $\mathrm{Vect}_{\mathbb{K}}^{\dagger}$ is not strict! However, we have a so-called **MacLane's strictification theorem**, which says that,

Each (non-strict) monoidal category is monoidal equivalent to a strict monoidal category.

Here, we have a strict monoidal category $\overline{\mathrm{Vect}_{\mathbb{K}}}$:

- Obj.: $[n]$, $n \in \mathbb{N}_{\geq 0}$, $I = [0]$.
- Mor.: $\overline{\mathrm{Vect}_{\mathbb{K}}}([m],[n]) = M_{n\times m}(\mathbb{K})$.
- $\otimes$: $[n]\otimes[m] := [mn]$ $\qquad A\otimes B = (a_{ij}B)$ Kronecker product

However, there is a canonical embedding $\overline{\mathrm{Vect}_{\mathbb{K}}} \hookrightarrow \mathrm{Vect}_{\mathbb{K}}^{\dagger}$. via $[n] \longmapsto \mathbb{K}^n$, which is a monoidal equivalence.

Hence, our operator invariants are induced by strict monoidal functor,

$$\begin{tikzcd} & \overline{\mathrm{Vect}_{\mathbb{K}}} \arrow[d, hook] \\ \mathcal{T} \arrow[ur] \arrow[r, "Q"'] & \mathrm{Vect}_{\mathbb{K}}^{\dagger} \end{tikzcd}$$

and, up to monoidal isomorphisms, Q only depends on $\dim_{\mathbb{K}} V$, R, μ.

ii). Question by Songjin: Does the pair (α, β) contain more information than μ?

Answer: Up to monoidal equivalence, NO!

$\exists$ a monoidal natural morphism $\eta : Q_{V,V^*,R,\mathrm{id}_V,\mu} \longrightarrow Q_{V,W,R,\alpha,\beta}$ generated by
$$\begin{cases} \eta_+ = \mathrm{id}_V : V \to V \\ \eta_- = \alpha^* : V^* \to W. \end{cases}$$

η is a natural isomorphism by the rigidity condition.

<u>Aim for today</u>. Provide a 'more natural' construction for operator invariants of (colored / framed) tangles via ribbon Hopf algebras. $\Longrightarrow$ Reshetikhin - Turaev invariants.

<u>Question</u> Why do we need ribbon Hopf algebras?

- $\mathrm{Rep}^{f}_{\mathbb{K}}(H)$ is a ribbon category if H is a ribbon Hopf algebra.

- $f\mathcal{T}$, category of framed tangles, is a free ribbon category, which means that $\forall$ ribbon category $\mathcal{C}$ and $V \in \mathcal{C}$,
$$f\mathcal{T} \xrightarrow{\exists!\, Q} \mathcal{C}$$
s.t. Q is a braided monoidal functor, and
$$Q(+) = V.$$

- Combine the above two, $\forall$ ribbon Hopf algebra H,

$$\begin{tikzcd} f\mathcal{T} \arrow[r, "\exists!\, Q"] \arrow[dr] & \mathrm{Rep}^{f}_{\mathbb{K}}(H) \arrow[d, "F"] \\ & \mathrm{Vect}^{f}_{\mathbb{K}} \end{tikzcd}$$

s.t. $F \circ Q$ is an operator invariant as before.

<u>Outline</u>.

§1. Monoidal categories and monoidal functors

§2. Braiding, duality and pivot

§3. Ribbon category and operator invariants

§1. Monoidal categories and monoidal functors

Def 1.1 (Monoidal categories)

A monoidal category (or say, a tensor category) is a sextuple $(\mathcal{C}, \otimes, I, a, l, r)$ consisting of,

- a category $\mathcal{C}$,
- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, the tensor product,
- an object $I \in \mathcal{C}$, the unit object,
- a natural isomorphism $a : (-\otimes-)\otimes- \longrightarrow -\otimes(-\otimes-)$, the associator constraint,
- natural isomorphisms $l : I\otimes - \longrightarrow id_{\mathcal{C}}$, $r : -\otimes I \longrightarrow id_{\mathcal{C}}$, the unit constraints,

satisfying the following two axioms:

- (Pentagon axiom) $\forall\, U, V, W, X \in \mathcal{C}$,

$$\begin{tikzcd}
 & (U\otimes(V\otimes W))\otimes X \arrow[r, "a_{U,V\otimes W,X}"] & U\otimes((V\otimes W)\otimes X) \arrow[d, "id_U\otimes a_{V,W,X}"] \\
((U\otimes V)\otimes W)\otimes X \arrow[ur, "a_{U,V,W}\otimes id_X"] \arrow[dr, "a_{U\otimes V,W,X}"'] & & U\otimes(V\otimes(W\otimes X)) \\
 & (U\otimes V)\otimes(W\otimes X) \arrow[ur, "a_{U,V,W\otimes X}"'] &
\end{tikzcd}$$

- (triangle axiom) $\forall\, V, W \in \mathcal{C}$,

$$\begin{tikzcd}
(V\otimes I)\otimes W \arrow[rr, "a_{V,I,W}"] \arrow[dr, "r_V\otimes id_W"'] & & V\otimes(I\otimes W) \arrow[dl, "id_V\otimes l_W"] \\
 & V\otimes W &
\end{tikzcd}$$

Example.

1). $(\mathcal{C} = \mathrm{Vect}_{\mathbb{k}}, \otimes_{\mathbb{k}}, \mathbb{k})$, category of vector spaces over a field $\mathbb{k}$ is a monoidal category.

$\mathrm{Vect}^{f}_{\mathbb{k}} \hookrightarrow \mathrm{Vect}_{\mathbb{k}}$, subcategory of finite dimensional vector spaces.

2). $(\mathcal{C} = \mathrm{Rep}_{\mathbb{k}}(G), \otimes_{\mathbb{k}}, \mathbb{k})$, category of representations of a group G

(i.e., subcategory of $\mathrm{Vect}_{\mathbb{k}}$, whose objects are $\mathbb{k}G$-modules with $\mathbb{k}G$-module homomorphisms between them as morphisms)

Notice that, the G-action on tensor products is defined as,

$$g.(v\otimes w) := g.v \otimes g.w, \quad \forall\, g\in G,\ v\in V,\ w\in W,\ V, W\in \mathrm{Rep}_{\mathbb{k}}(G)$$

and $\mathbb{k}$ is viewed as a trivial representation of G.

$$g.\lambda = \lambda \qquad \forall\, g\in G,\ \lambda\in\mathbb{k}.$$

3) In fact, any category $\mathcal{C}$ with finite (co)products is a monoidal category, for example,

- $(\underline{Set}, \times, \{*\})$, $(\underline{Set}, \sqcup, \emptyset)$
- $(\underline{Grp}, \times, \langle 1\rangle)$, $(\underline{Grp}, *, \langle 1\rangle)$
- $(\underline{Ab}, \times, \langle 1\rangle)$, $(\underline{Ab}, \oplus, \langle 1\rangle)$.

4). In general, for any 2-category $\mathcal{C}$ and any object $C\in\mathcal{C}$, $(\mathrm{End}_{\mathcal{C}}(C), \circ, \mathrm{id}_C)$ is a monoidal category.

For example, $\mathcal{C} = \underline{Cat} \ni \mathcal{D}$, $\mathrm{End}(\mathcal{D})$ is the category of all endofunctors. $(\mathrm{End}(\mathcal{D}), \circ, \mathrm{Id}_{\mathcal{D}})$ is a monoidal functor.

<u>Prop 1.2</u> $(B, m, \Delta, \eta, \varepsilon)$ be a bialgebra over $\Bbbk$, then $(\mathrm{Rep}_{\Bbbk}(B), \otimes_{\Bbbk}, \Bbbk)$ is a monoidal category. Here, $\mathrm{Rep}_{\Bbbk}(B)$ is the category of all $\Bbbk$-representation of B, regarded as a subcategory of $\mathrm{Vect}_{\Bbbk}$.

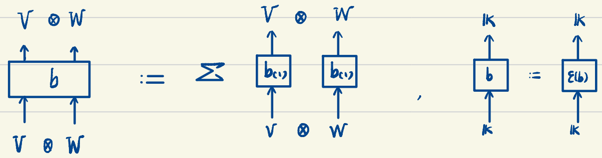
<u>Pf.</u> $\forall V, W\in \mathrm{Rep}_{\Bbbk}(B)$, $\forall v\in V$, $w\in W$, $b\in B$, define the action of b on $V\otimes W$ by

$$b.(v\otimes w) := \sum b_{(1)}.v \otimes b_{(2)}.w$$

Here, $\Delta b = \sum b_{(1)}\otimes b_{(2)}$ is the Sweedler notation.

Define the action of b on $\Bbbk$ by,

$$b.\lambda := \varepsilon(b)\lambda.$$



- Δ is an algebra homomorphism $\iff$ $V\otimes W$ is a B-mod.
- $(\Delta\otimes id)\circ\Delta = (id\otimes\Delta)\circ\Delta \iff a_{U,V,W}: (U\otimes V)\otimes W \longrightarrow U\otimes(V\otimes W)$ is B-equivariant
- $(\varepsilon\otimes id)\circ\Delta = (id\otimes\varepsilon)\circ\Delta = id \iff l_V: \Bbbk\otimes V\to V$ and $r_V: V\otimes \Bbbk\to V$ is B-equivariant.

$\square$

<u>Rem.</u> Similarly, $\mathrm{Rep}^{f}_{\Bbbk}(B)$, category of finite dimensional $\Bbbk$-representations of a bialgebra B, is also a monoidal category.

Def 1.3. A monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$ is called strict if a, l, r are all identity maps.

Example.

1). $\overline{Vect}_{\mathbb{K}}$

- Obj: $[n]$, $n \in \mathbb{N}_{\geq 0}$, $I = [0]$.
- Mor.: $\overline{Vect}_{\mathbb{K}}([m], [n]) = M_{n\times m}(\mathbb{K})$.
- $\otimes$: $[n] \otimes [m] := [mn]$ $\quad A \otimes B$ Kronecker product

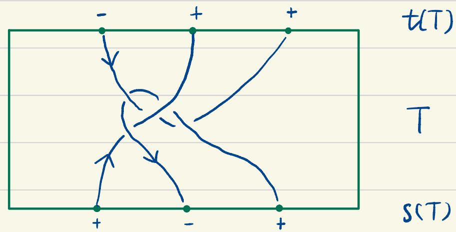
2). $\mathcal{B}$, the braid category

- Obj: $[n]$, $n \in \mathbb{N}_{\geq 0}$, $I = [0]$.
- Mor.: $\mathcal{B}([m], [n]) = \begin{cases} B_n & \text{if } m = n \\ \phi & \text{else} \end{cases}$
- $\otimes$: $[n] \otimes [m] := [m+n]$

3). $(\mathcal{T}, \otimes, I)$, the tangle category

- Obj: total ordered set of "$\pm$"-colored points

  e.g. $A = \{+, -, +\}$ $\quad B = \{-, +, +\}$
- Mor.: equivalent classes of tangles, $[T] \in \mathcal{T}(A, B)$ if $s(T) = A$, $t(T) = B$.
- I : identity object given by the empty set. $I = \phi$.

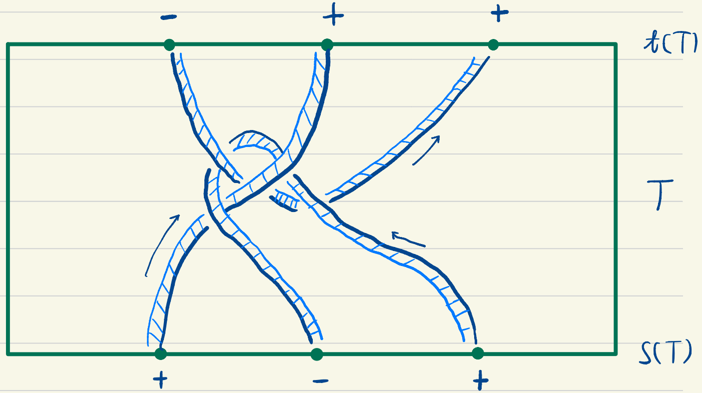


4). $f\mathcal{T}$, the framed tangle category

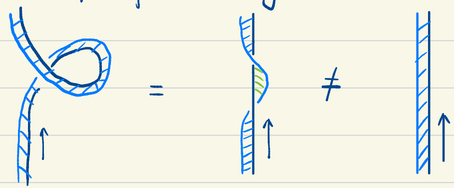
- Obj. : total ordered set of "$\pm$"-colored points.
- Mor. : equivalent classes of tangles with 'blackboard framing'

$[T] \in f\mathcal{T}(A, B)$ if $s(T) = A$, $t(T) = B$.

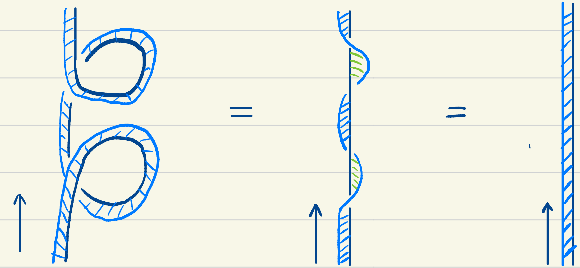
- I : identity object given by the empty set.



Notice that for framed tanges,



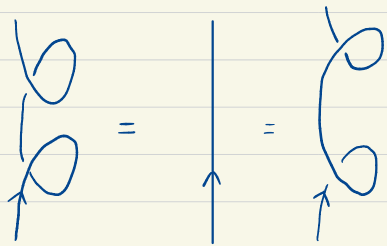
However,



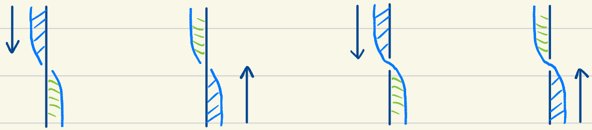
- Framed tangles = tangles + {# of full twist of each component}.

- $\{\text{framed tangles}\}/\text{equivalence} \longleftrightarrow \dfrac{\{\text{tangle diagrams}\}/\text{isotopy of } \mathbb{R}\times I}{\text{framed Riedemeister moves}}$ (fRI , RII , RIII)

(fRI)



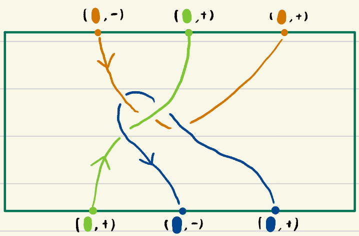
Rem. Since we only use blackboard framing, all the twists are integers, which means that framed tangles with 'half twisted' components are not included.



5). Colored (framed) tangles category, C-$\mathcal{T}$ (C-$f\mathcal{T}$).

Given a finite set C, 'the set of colors'. C-$\mathcal{T}$ consists of

- Obj. : total ordered set of $C\times\{\pm\}$-colored points
- Mor. : equivalent classes of C-colored tangles, s.t., each component is assigned a color in C and colors of intervals are compatible with colors of boundary points.



Def 1.4 (monoidal functor)

1). A monoidal functor from $(\mathcal{C}, \otimes_\mathcal{C}, I_\mathcal{C}, a^\mathcal{C}, l^\mathcal{C}, r^\mathcal{C})$ to $(\mathcal{D}, \otimes_\mathcal{D}, I_\mathcal{D}, a^\mathcal{D}, l^\mathcal{D}, r^\mathcal{D})$ is a triple (F, J), s.t.

- $F : \mathcal{C} \to \mathcal{D}$ is a functor, such that $F(I_\mathcal{C}) \cong I_\mathcal{D}$.
- $J : F(-)\otimes_\mathcal{D} F(-) \longrightarrow F(-\otimes_\mathcal{C} -)$ is a natural isomorphism, satisfies

(monoidal structure axiom)

$$\begin{tikzcd}
(F(U)\otimes_\mathcal{D} F(V))\otimes_\mathcal{D} F(W) \arrow[r, "J_{U,V}\otimes_\mathcal{D} id_{F(W)}"] \arrow[d, "a^\mathcal{D}_{F(U),F(V),F(W)}"'] & F(U\otimes_\mathcal{C} V)\otimes_\mathcal{D} F(W) \arrow[d, "J_{U\otimes_\mathcal{C} V, W}"'] \\
F(U)\otimes_\mathcal{D}(F(V)\otimes_\mathcal{D} F(W)) \arrow[d, "id_{F(U)}\otimes_\mathcal{D} J_{U,V}"'] & F((U\otimes_\mathcal{C} V)\otimes_\mathcal{C} W) \arrow[d, "F(a^\mathcal{C}_{U,V,W})"'] \\
F(U)\otimes_\mathcal{D}(F(V\otimes_\mathcal{C} W)) \arrow[r, "J_{U,V\otimes_\mathcal{C} W}"'] & F(U\otimes_\mathcal{C}(V\otimes_\mathcal{C} W))
\end{tikzcd}$$

2). A monoidal functor (F, J) is called strict, if J is an identity map, i.e., $\forall U, V \in \mathcal{C}$,

$$F(U\otimes_\mathcal{C} V) = F(U)\otimes_\mathcal{D} F(V).$$

3). (F, J) is called a monoidal equivalence, if F is an equivalence between categories.

4). A monoidal natural morphism $\eta : (F^1, J^1) \to (F^2, J^2)$ between monoidal functors from $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, a^{\mathcal{C}}, l^{\mathcal{C}}, r^{\mathcal{C}})$ to $(\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, a^{\mathcal{D}}, l^{\mathcal{D}}, r^{\mathcal{D}})$ is a natural morphism $\eta : F^1 \longrightarrow F^2$ such that $\eta_{I_{\mathcal{C}}}$ is an isomorphism and

$$\begin{tikzcd}
F^1(U) \otimes_{\mathcal{D}} F^1(V) \arrow[r, "\eta_U \otimes \eta_V"] \arrow[d, "J^1_{U,V}"'] & F^2(U) \otimes_{\mathcal{D}} F^2(V) \arrow[d, "J^2_{U,V}"] \\
F^1(U \otimes_{\mathcal{C}} V) \arrow[r, "\eta_{U \otimes_{\mathcal{C}} V}"] & F^2(U \otimes_{\mathcal{C}} V)
\end{tikzcd}$$

Example.

1). Forgetful functors, e.g.

$$F : \mathrm{Rep}_{\mathbb{K}}(B) \longrightarrow \mathrm{Vect}_{\mathbb{K}}$$
$$\mathrm{Rep}^f_{\mathbb{K}}(B) \longrightarrow \mathrm{Vect}^f_{\mathbb{K}}.$$

2).
$$C\text{-}fT \longrightarrow fT \quad \text{forget colors}$$
$$fT \longrightarrow T \quad \text{forget framing}$$
$$\mathcal{B} \hookrightarrow T$$

3).
$$\overline{\mathrm{Vect}_{\mathbb{K}}} \hookrightarrow \mathrm{Vect}^f_{\mathbb{K}}$$
$$[n] \longmapsto \mathbb{K}^n$$

This inclusion is indeed a monoidal equivalence.

Thm 1.5 (MacLane strictness theorem)

Any monoidal category is monoidally equivalent to a strict monoidal category.

Coro 1.6. (MacLane coherence theorem)

Suppose $f, g : U \to V$ are two isomorphisms generated by associativity and unit constraints, their inverses and identity morphisms via composition and tensor product, then $f = g$.

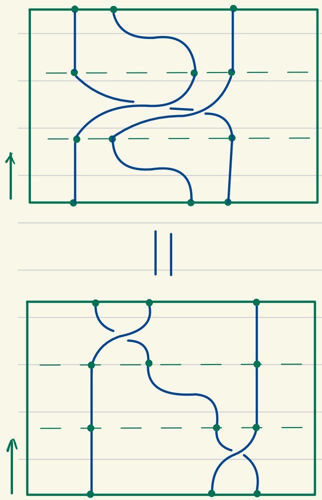
§2. Braiding, duality and pivot

Def 2.1.

1). For a monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$, a **braiding** is a natural isomorphism $c : - \otimes - \longrightarrow - \otimes^{op} -$, satisfies (hexagon axioms). $\forall U, V, W \in \mathcal{C}$,

- $$\begin{tikzcd} (U\otimes V)\otimes W \arrow[r, "c_{U,V}\otimes id_W"] \arrow[d, "a_{U,V,W}"'] & (V\otimes U)\otimes W \arrow[d, "a_{V,U,W}"] \\ U\otimes(V\otimes W) \arrow[d, "c_{U,V\otimes W}"'] & V\otimes(U\otimes W) \arrow[d, "id_V\otimes c_{U,W}"] \\ (V\otimes W)\otimes U \arrow[r, "a_{V,W,U}"] & V\otimes(W\otimes U) \end{tikzcd}$$

- $$\begin{tikzcd} U\otimes(V\otimes W) \arrow[r, "id_U\otimes c_{V,W}"] \arrow[d, "a^{-1}_{U,V,W}"'] & U\otimes(W\otimes V) \arrow[d, "a^{-1}_{U,W,V}"] \\ (U\otimes V)\otimes W \arrow[d, "c_{U\otimes V,W}"'] & (U\otimes W)\otimes V \arrow[d, "c_{U,W}\otimes id_V"] \\ W\otimes(U\otimes V) \arrow[r, "a^{-1}_{W,U,V}"] & (W\otimes U)\otimes V \end{tikzcd}$$



2). A monoidal category with a braiding is called a **braided monoidal category**.

3). A braiding c is called symmetric if

$$c_{W,V} = (c_{V,W})^{-1}$$

A braided monoidal category with a symmetric braiding is called a **symmetric monoidal category**.

Prop 2.2.

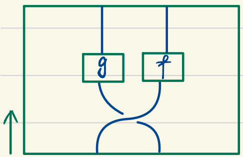
1). If the braided monoidal category is strict, then the Hexagon axioms can be reduced to equations,

$$c_{U\otimes V,W} = (c_{U,W}\otimes id_V)\circ(id_U\otimes c_{V,W})$$
$$c_{U,V\otimes W} = (id_V\otimes c_{U,W})\circ(c_{U,V}\otimes id_W)$$

2). By the naturality of $c: -\otimes- \longrightarrow -\otimes^{op}-$,

$\forall f\in\mathcal{C}(U_1,U_2),\ g\in\mathcal{C}(V_1,V_2)$, the following equation holds

$$(g\otimes f)\circ c_{U_1,V_1} = c_{U_2,V_2}\circ(f\otimes g)$$



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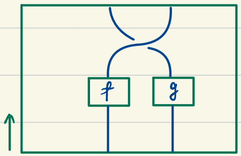


3). Combine Hexagon axioms, coherence theorem and 2)., we have

$$\begin{tikzcd}
(V\otimes I)\otimes W \arrow[rr, "a_{V,I,W}"] \arrow[dd, "c_{V,I}\otimes id_W"'] \arrow[dr, "r_V\otimes id_W"] & & V\otimes(I\otimes W) \arrow[rr, "c_{V,I\otimes W}"] \arrow[dl, "id_V\otimes l_W"] & & (I\otimes W)\otimes V \arrow[dd, "a_{I,W,V}"] \arrow[dl, "l_W\otimes id_V"'] \\
 & V\otimes W \arrow[rr, "c_{V,W}"] & & W\otimes V & \\
(I\otimes V)\otimes W \arrow[rr, "a_{I,V,W}"] \arrow[ur, "l_V\otimes id_W"] & & I\otimes(V\otimes W) \arrow[rr, "id_I\otimes c_{V,W}"] \arrow[ul, "l_{V\otimes W}"'] & & I\otimes(W\otimes V) \arrow[ul, "l_{W\otimes V}"']
\end{tikzcd}$$

G

$$\Longrightarrow \quad \begin{tikzcd}
(V\otimes I)\otimes W \arrow[rr, "c_{V,I}\otimes id_W"] \arrow[dr, "r_V\otimes id_W"'] & & (I\otimes V)\otimes W \arrow[dl, "l_V\otimes id_W"] \\
 & V\otimes W &
\end{tikzcd}$$

$$\Longrightarrow \quad c_{V,I} = l_V^{-1}\circ r_V$$

In cases $\mathcal{C}$ is strict, $c_{V,I} = id_V$.

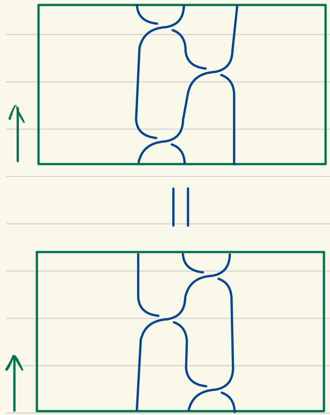
4). If c is a braiding, then its conjugate $\bar{c}$ defind by
$$\bar{c}_{V,W} := (c_{W,V})^{-1} : V\otimes W \longrightarrow W\otimes V$$
is another braiding. $c=\bar{c}$ iff c is symmetric.

5). Combine two Hexagon axioms and naturality of c,

$$\begin{tikzcd}
 & (V\otimes U)\otimes W \arrow[rr] & & V\otimes(U\otimes W) \arrow[dr, "id_V\otimes c_{U,W}"] & \\
(U\otimes V)\otimes W \arrow[ur, "c_{U,V}\otimes id_W"] \arrow[dr] & & & & V\otimes(W\otimes U) \\
 & U\otimes(V\otimes W) \arrow[rr, "c_{U,V\otimes W}"] \arrow[d, "id_U\otimes c_{V,W}"'] & & (V\otimes W)\otimes U \arrow[ur] \arrow[d, "c_{V,W}\otimes id_U"] & \\
 & U\otimes(W\otimes V) \arrow[rr, "c_{U,W\otimes V}"] & & (W\otimes V)\otimes U \arrow[dr] & \\
(U\otimes W)\otimes V \arrow[ur] \arrow[dr, "c_{U,W}\otimes id_V"'] & & & & W\otimes(V\otimes U) \\
 & (W\otimes U)\otimes V \arrow[rr] & & W\otimes(U\otimes V) \arrow[ur, "id_W\otimes c_{U,V}"'] &
\end{tikzcd}$$

In cases $\mathcal{C}$ is strict, we have

$$\begin{aligned}
&(c_{V,W}\otimes id_U)\circ(id_V\otimes c_{U,W})\circ(c_{U,V}\otimes id_W)\\
=&(id_W\otimes c_{U,V})\circ(c_{U,W}\otimes id_V)\circ(id_U\otimes c_{V,W})
\end{aligned}$$



Example.

1). Any monoidal category arises from finite (co)products is naturally a symmetric monoidal category.

2). $\mathrm{Vect}_{\mathbb{k}}$ is a symmetric monoidal category with a braiding

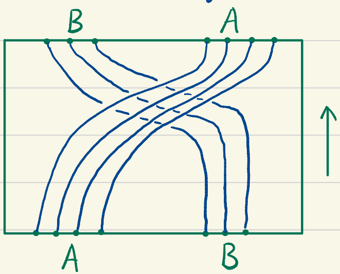
$$\tau_{V,W}: V\otimes W \longrightarrow W\otimes V.$$

However, $\mathrm{Rep}_{\mathbb{k}}(B)$ for a bialgebra B is not symmetric in general. The restriction of τ in $\mathrm{Rep}_{\mathbb{k}}(B)$ is defined iff B is cocommutative.

Prop 2.3. $\mathcal{B}$, $\mathcal{T}$, $f\mathcal{T}$, $C\text{-}\mathcal{T}$, $C\text{-}f\mathcal{T}$ are all strict braided monoidal categories.

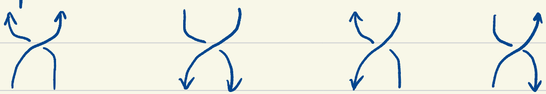
pf. We prove it for $\mathcal{T}$, others are similar.

For any total ordered sets of '$\pm$' points, A and B, define $C_{A,B}$ to be the equivalent class of a tangle



- Orientation for each strand is determined by the sign of its boundary points.
- All the crossings above are  .

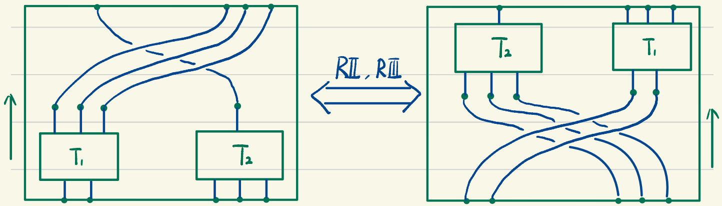
For example,



$C_{+,+}$ $\quad$ $C_{-,-}$ $\quad$ $C_{-,+}$ $\quad$ $C_{+,-}$

We can show that

1). C is natural.



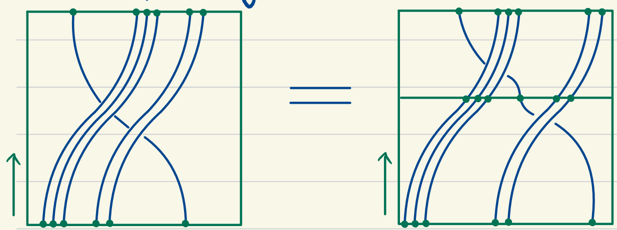
2). C is a natural isomorphism.

$(C_{A,B})^{-1}$ is defind by replacing all  in $C_{B,A}$ by  .

$$C_{A,B} \circ (C_{A,B})^{-1} \overset{R\text{II}}{\Longleftrightarrow} id_B \otimes id_A = id_{B\otimes A}$$

$$(C_{A,B})^{-1} \circ C_{A,B} \overset{R\text{II}}{\Longleftrightarrow} id_A \otimes id_B = id_{A\otimes B}.$$

3). C satisfies Hexagon axioms.



$\square$

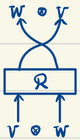
Prop 2.4. Let B be a bialgebra, $\mathrm{Rep}_{\mathbb{k}}(B)$ is a braided monoidal category iff B allows a quasitriangular structure.

pf. • 'If' part.

(B, R) is a quasi-triangular bialgebra, then $\forall V, W \in \mathrm{Rep}_{\mathbb{k}}(B)$, define a $\mathbb{k}$-linear map

$$c_{V,W} : V \otimes W \longrightarrow W \otimes V$$
$$v \otimes w \longmapsto \sum R^{(2)}.w \otimes R^{(1)}.v$$

Here, $R = \sum R^{(1)} \otimes R^{(2)}$. In graphic expression,



$=: \; c_{V,W}$

One can show that:

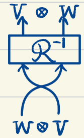
1). $c_{V,W}$ is B-equivariant. $\Leftarrow \tau(\Delta b)\cdot R = R \cdot \Delta b$

$$\begin{aligned} c_{V,W}(b.(v\otimes w)) &= \sum R^{(2)} b_{(2)}.w \otimes R^{(1)} b_{(1)}.v \\ &= \sum b_{(1)} R^{(2)}.w \otimes b_{(2)} R^{(1)}.v \\ &= b.(c_{V,W}(v\otimes w)). \end{aligned}$$

2). $c_{V,W}$ is an isomorphism. $\Leftarrow R$ is invertible.

$$(c_{V,W})^{-1} : W \otimes V \longrightarrow V \otimes W$$
$$w \otimes v \longmapsto \sum (R^{-1})^{(1)}.v \otimes (R^{-1})^{(2)}.w$$

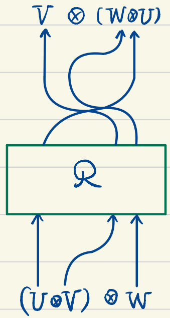
Here, $R^{-1} = \sum (R^{-1})^{(1)} \otimes (R^{-1})^{(2)}$. In graphic expression,



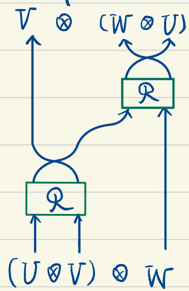
$= (c_{V,W})^{-1}$.

3). $c : - \otimes - \longrightarrow - \otimes^{op} -$ is natural.

4). c satisfies Hexagon axioms. $\Leftarrow \begin{cases} (\Delta \otimes id) R = R_{13} R_{23} \\ (id \otimes \Delta) R = R_{13} R_{12} \end{cases}$



$=$



Hence, $\mathrm{Rep}_{\mathbb{k}}(B)$ has a braiding c on it.

- "Only if" part.

$c : - \otimes - \longrightarrow - \otimes^{op} -$ is a braiding on $\mathrm{Rep}_{\mathbb{K}}(B)$, then since $B \in \mathrm{Rep}_{\mathbb{K}}(B)$, define

$$\mathcal{R} := \tau(c_{B,B}(\eta(1) \otimes \eta(1))) \in B \otimes B$$

One can show that:

1). $\mathcal{R}$ is invertible. $\Leftarrow$ $c_{B,B}$ is invertible.

$$\mathcal{R}^{-1} := \tau(c_{B,B}^{-1}(\eta(1) \otimes \eta(1))) \in B \otimes B$$

2). $\forall b \in B, \tau(\Delta b) \cdot \mathcal{R} = \mathcal{R} \cdot \Delta b \Leftarrow \begin{cases} c_{B,B} \text{ is } B\text{-equivariant}. \\ c \text{ is natural}. \end{cases}$

Define $\begin{aligned} R_b : B &\longrightarrow B \\ b' &\longmapsto b'b \end{aligned}$. Then R_b is B-equivariant.

$$\begin{aligned} \tau(\Delta b) \cdot \mathcal{R} &= \tau(b . c_{B,B}(\eta(1) \otimes \eta(1))) \\ &= \tau(c_{B,B}(\Delta b)) \\ &= \tau(c_{B,B}(R_{b_{(1)}}(\eta(1)) \otimes R_{b_{(2)}}(\eta(1)))) \\ &= (R_{b_{(1)}} \otimes R_{b_{(2)}})(\tau(c_{B,B}(\eta(1) \otimes \eta(1)))) \\ &= \mathcal{R} \cdot \Delta b. \end{aligned}$$

3). $\begin{cases} (\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \\ (id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \end{cases} \Leftarrow \begin{cases} \text{Hexagon axioms.} \\ c \text{ is natural} \end{cases}$

Notice that $\Delta : B \longrightarrow B \otimes B$ is B-equivariant.

$$\begin{aligned} &(\Delta \otimes id)\mathcal{R} \\ = &(\Delta \otimes id) \circ \tau(c_{B,B}(\eta(1) \otimes \eta(1))) \\ = &\tau(c_{B\otimes B,B} \circ (\Delta \otimes id)(\eta(1) \otimes \eta(1))) \\ = &\tau(c_{B\otimes B,B}((\eta(1) \otimes \eta(1)) \otimes \eta(1))) \\ = &\tau(a_{B,B,B} \circ (c_{B,B} \otimes id) \circ a_{B,B,B}^{-1} \circ (id \otimes c_{B,B})(\eta(1) \otimes (\eta(1) \otimes \eta(1)))) \\ = &\tau(a_{B,B,B} \circ (c_{B,B} \otimes id)(\Sigma(\eta(1) \otimes \mathcal{R}_1^{(2)}) \otimes \mathcal{R}_1^{(1)})) \\ = &\Sigma\, \tau(a_{B,B,B} \circ (c_{B,B} \otimes id) \circ ((id \otimes R_{\mathcal{R}_1^{(2)}}) \otimes R_{\mathcal{R}_1^{(1)}})((\eta(1) \otimes \eta(1)) \otimes \eta(1))) \\ = &\Sigma\, \tau((R_{\mathcal{R}_1^{(2)}} \otimes (id \otimes R_{\mathcal{R}_1^{(1)}})) \circ a_{B,B,B} \circ (c_{B,B} \otimes id)((\eta(1) \otimes \eta(1)) \otimes \eta(1))) \\ = &\Sigma\, \tau((R_{\mathcal{R}_1^{(2)}} \otimes (id \otimes R_{\mathcal{R}_1^{(1)}}))(\mathcal{R}_2^{(2)} \otimes (\mathcal{R}_2^{(1)} \otimes \eta(1)))) \\ = &\Sigma\, \tau(\mathcal{R}_2^{(2)}\mathcal{R}_1^{(2)} \otimes (\mathcal{R}_2^{(1)} \otimes \mathcal{R}_1^{(1)})) \\ = &\mathcal{R}_{13}\mathcal{R}_{23}. \end{aligned}$$

The other one is similar. $\square$

Rem.

1). If $\mathrm{Rep}_{\mathbb{k}}(B)$ is replaced by $\mathrm{Rep}^{f}_{\mathbb{k}}(B)$, the "If" part still holds; but "Only if" part doesn't holds in general.

2). We call a quasitriangular bialgebra $(B, \mathcal{R})$ **triangular** if $\mathcal{R}_{21}\mathcal{R} = \eta(1) \otimes \eta(1)$. Then $(B, \mathcal{R})$ is triangular iff $\mathrm{Rep}_{\mathbb{k}}(B)$ is symmetric.

Def 2.5. Let $\mathcal{C}$ and $\mathcal{D}$ be braided monoidal categories.

1). A monoidal functor $(F, J): \mathcal{C} \longrightarrow \mathcal{D}$ is called a **braided monoidal functor** if $\forall V, W \in \mathcal{C}$,

$$\begin{tikzcd}
F(V) \otimes_{\mathcal{D}} F(W) \arrow[r, "c^{\mathcal{D}}_{F(V),F(W)}"] \arrow[d, "J_{V,W}"'] & F(W) \otimes_{\mathcal{D}} F(V) \arrow[d, "J_{W,V}"] \\
F(V \otimes_{\mathcal{C}} W) \arrow[r, "F(c^{\mathcal{C}}_{V,W})"] & F(W \otimes_{\mathcal{C}} V)
\end{tikzcd}$$

2). Braided monoidal functors between symmetric monoidal categories are called **symmetric monoidal functor**.

Rem.

By MacLane strictness theorem, each braided monoidal category is equivalent to a strict one by a braided monoidal equivalence.

Prop 2.6. $\mathcal{B}$ is a free strict braided monoidal category generated by one object $[1]$ and one morphism, the braiding $c_{[1],[1]}$. In other words, for any strict braided monoidal category $\mathcal{C}$ and any $V \in \mathcal{C}$, there exists a unique strict braided monoidal functor

$$Q: \mathcal{B} \longrightarrow \mathcal{C}$$

such that $Q([1]) = V$.

Rem.

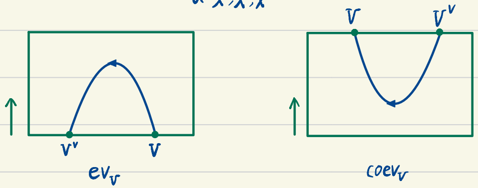
Similar as $\mathcal{B}$, we can consider $\mathcal{S}$ by replacing B_n to symmetric group S_n. Then $\mathcal{S}$ is a free strict symmetric monoidal category.

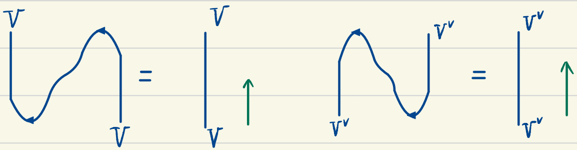
Def 2.7. Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a monoidal category,

1). for an object $X \in \mathcal{C}$, $X^{\vee} \in \mathcal{C}$ is called a right dual of X with a right evaluation morphism $ev_X : X^{\vee} \otimes X \to I$, and a right coevaluation morphism $coev_X : I \to X \otimes X^{\vee}$, if

$$\begin{tikzcd}
X \arrow[r, "id_X"] \arrow[d, "coev_X \otimes id_X"'] & X \\
(X \otimes X^{\vee}) \otimes X \arrow[r, "a_{X,X^{\vee},X}"'] & X \otimes (X^{\vee} \otimes X) \arrow[u, "id_X \otimes ev_X"']
\end{tikzcd}$$

$$\begin{tikzcd}
X^{\vee} \arrow[r, "id_{X^{\vee}}"] \arrow[d, "id_{X^{\vee}} \otimes coev_X"'] & X^{\vee} \\
X^{\vee} \otimes (X \otimes X^{\vee}) \arrow[r, "a^{-1}_{X^{\vee},X,X^{\vee}}"'] & (X^{\vee} \otimes X) \otimes X^{\vee} \arrow[u, "ev_X \otimes id_{X^{\vee}}"']
\end{tikzcd}$$

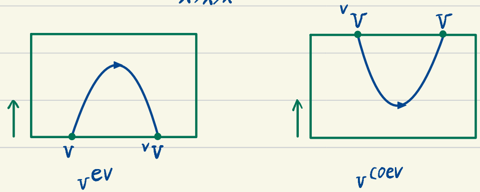




2). for an object $X \in \mathcal{C}$, ${}^{\vee}X \in \mathcal{C}$ is called a left dual of X with a left evaluation morphism ${}_X ev : X \otimes {}^{\vee}X \to I$, and a left coevaluation morphism ${}_X coev : I \to {}^{\vee}X \otimes X$, if

$$\begin{tikzcd}
X \arrow[r, "id_X"] \arrow[d, "id_X \otimes {}_X coev"'] & X \\
X \otimes ({}^{\vee}X \otimes X) \arrow[r, "a^{-1}_{X,{}^{\vee}X,X}"'] & (X \otimes {}^{\vee}X) \otimes X \arrow[u, "{}_X ev \otimes id_X"']
\end{tikzcd}$$

$$\begin{tikzcd}
{}^{\vee}X \arrow[r, "id_{{}^{\vee}X}"] \arrow[d, "{}_X coev \otimes id_{{}^{\vee}X}"'] & {}^{\vee}X \\
({}^{\vee}X \otimes X) \otimes {}^{\vee}X \arrow[r, "a_{{}^{\vee}X,X,{}^{\vee}X}"'] & {}^{\vee}X \otimes (X \otimes {}^{\vee}X) \arrow[u, "id_{{}^{\vee}X} \otimes {}_X ev"']
\end{tikzcd}$$



3). $\mathcal{C}$ is called right / left rigid, if each object in $\mathcal{C}$ has a right / left dual, and is called rigid, if $\mathcal{C}$ is both right and left rigid.

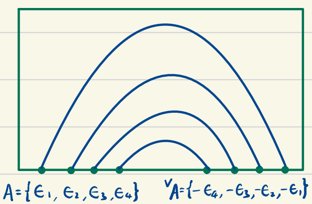
Example.

1). $\mathrm{Vect}_{\mathbb{K}}^{f}$ is rigid. $\forall V \in \mathrm{Vect}_{\mathbb{K}}^{f}$,

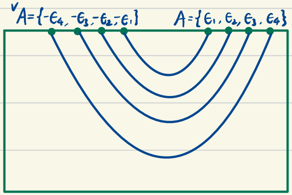
- $V^{\vee} = {}^{\vee}V = V^{*} := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$.
- $ev_V(f \otimes v) = {}_V ev(v \otimes f) = f(v)$, $\forall v \in V, f \in V^*$
- $coev_V(\lambda) = \sum_i \lambda v_i \otimes v^i$, ${}_V coev(\lambda) = \sum_i \lambda v^i \otimes v_i$

$\forall \lambda \in \mathbb{K}$. Here $\{v_i\}$ is a basis of V, $\{v^i\}$ is the dual basis.

2). $\mathcal{T}$, $f\mathcal{T}$, C-$\mathcal{T}$, C-$f\mathcal{T}$ is rigid.



A^{ev}



A^{coev}

- $\epsilon_i \in \{+, -\}$, orientation of each strand is determined by boundary points.
- For any object A, $A^{\vee} = {}^{\vee}A$.

Prop 2.8.

Let H be a Hopf algebra over $\mathbb{K}$. Then $\mathrm{Rep}_{\mathbb{K}}^{f}(H)$ is a right rigid monoidal category. Especially, if the antipode S of H is invertible, then $\mathrm{Rep}_{\mathbb{K}}^{f}(H)$ is rigid.

pf: $\forall V \in \mathrm{Rep}_{\mathbb{K}}^{f}(H)$, take $V^{\vee} := V^{*}$. We will show that,

1). V^* is a H-mod. $\Longleftarrow$ S is an antihomomorphism.

The H-action on V^* is define as, $\forall h \in H, f \in V^*, \forall v \in V$

$$h.f(v) := f(S(h).v)$$

2). ev_V and $coev_V$ are H-equivariant. $\Longleftarrow \begin{cases} m\circ(S\otimes id)\circ\Delta = \eta(1)\varepsilon \\ m\circ(id\otimes S)\circ\Delta = \eta(1)\varepsilon \end{cases}$

- $\forall h \in H, f \in V^*, v \in V$,

$h.ev_V(f\otimes v) = \varepsilon(h)\, ev_V(f\otimes v) = ev_V(\varepsilon(h) f \otimes v)$

$= f(\varepsilon(h) v) = f(S(h_{(1)}) h_{(2)}.v) = ev_V(h_{(1)}.f \otimes h_{(2)}.v)$

$= ev_V(h.(f\otimes v))$.

- $\forall h \in H, \lambda \in \mathbb{K}$,

$h.coev_V(\lambda) = h.(\lambda v_i \otimes v^i) = \lambda\, h_{(1)}.v_i \otimes h_{(2)}.v^i$

$= \lambda (h_{(1)}.)_i^j v_j \otimes (S(h_{(2)}).)_k^i v^k = \lambda (h_{(1)} S(h_{(2)}).)_k^j v_j \otimes v^k = \varepsilon(h)\, coev_V(\lambda)$.

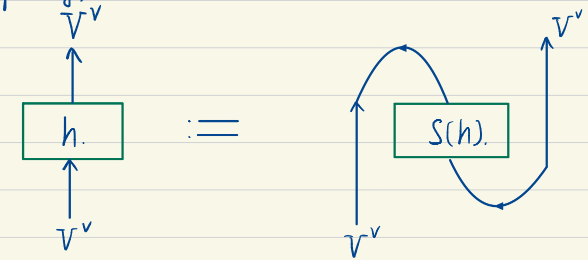
In cases where S is invertible, $\begin{cases} m^{op}\circ(S^{-1}\otimes id)\circ\Delta = \eta(1)\,\varepsilon, \\ m^{op}\circ(id\otimes S^{-1})\circ\Delta = \eta(1)\,\varepsilon. \end{cases}$

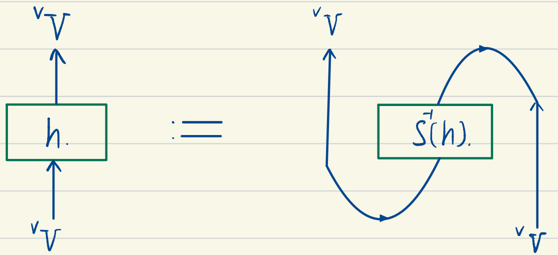
Hence, define ${}^{\vee}V = V^*$ with H-action

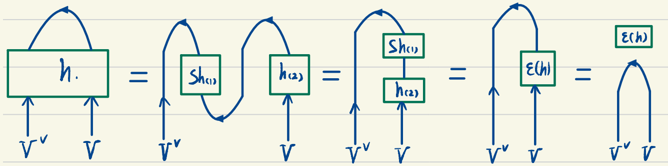
$$h.f := f\circ(S^{-1}(h).),$$

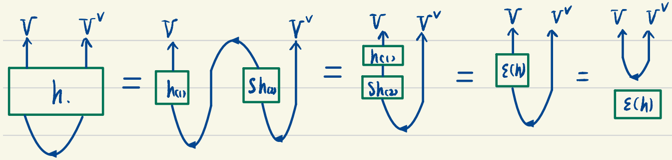
one can show that ${}_{V}ev$ and ${}_{V}coev$ are H-equivariant. $\square$

Graphically,









Rem.

The statement above can viewed as 'half' of Tannakian duality. The other half is also true when H is finite dimensional via Tannaka-Krein reconstruction method.

Prop 2.9. Let $\mathcal{C}$ and $\mathcal{D}$ be left (resp. right) rigid monoidal categories, $(F, J), (F', J') : \mathcal{C} \to \mathcal{D}$ be monoidal functors.

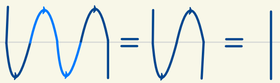
1). $\forall V \in \mathcal{C}$, left (resp. right) dual is unique up to isomorphism.

2). $\forall V \in \mathcal{C}$, $F({}^\vee V)$ (resp. $F(V^\vee)$) is left (resp. right) dual of $F(V)$.

3). $\eta : (F, J) \longrightarrow (F', J')$ is a monoidal natural morphism,

Then η is a monoidal natural isomorphism.

pf. 1).



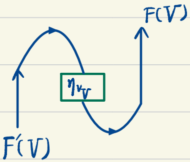
2). ${}_{F(V)}ev : F(V) \otimes_{\mathcal{D}} F({}^\vee V) \xrightarrow{J_{F(V), F({}^\vee V)}} F(V \otimes_{\mathcal{C}} {}^\vee V) \xrightarrow{F({}_V ev)} F(I_{\mathcal{C}}) \xrightarrow{\varphi} I_{\mathcal{D}}$

${}_{F(V)}coev : I_{\mathcal{D}} \xrightarrow{\varphi^{-1}} F(I_{\mathcal{C}}) \xrightarrow{F({}_V coev)} F({}^\vee V \otimes_{\mathcal{C}} V) \xrightarrow{J^{-1}_{F({}^\vee V), F(V)}} F({}^\vee V \otimes_{\mathcal{D}} V)$.

Here $\varphi : F(I_{\mathcal{C}}) \longrightarrow I_{\mathcal{D}}$ is a canonical isomorphism.

3). $\forall V \in \mathcal{C}$, define

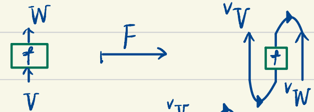
$\eta^{-1}_V :=$



$\square$

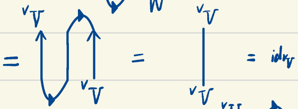
Prop 2.10. $(\mathcal{C}, \otimes, I, a, l, r)$ is left (resp. right) rigid, then ${}^\vee(-)$ (resp. $(-)^\vee$) $: (\mathcal{C}, \otimes, I, a, l, r) \longrightarrow (\mathcal{C}^{op}, \otimes^{op}, I, a, r^{-1}, l^{-1})$ gives a monoidal functor.

pf. Let F denote the corresponding functor. Define



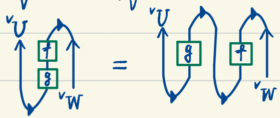
$=: F(f) \in \mathcal{C}^{op}({}^\vee V, {}^\vee W)$.

Then $F(id_V) =$



$= id_{{}^\vee V}$

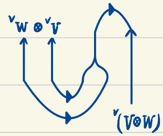
$F(f \circ g) =$



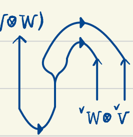
$= F(f) \circ F(g)$

Hence, F is a functor. Define

$J_{V,W} :=$



$J^{-1}_{V,W} :=$



One can show that:

1). J is a natural isomorphism.

2). (F, J) satisfies monoidal structure axiom.

3). $F(I) = {}^\vee I \cong I$.

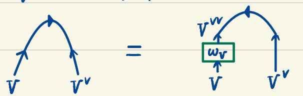
$\square$

Def 2.11. Let $\mathcal{C}$ be a right rigid monoidal category, a pivotal structure on $\mathcal{C}$ is a monoidal natural isomorphism $\omega : Id_{\mathcal{C}} \longrightarrow (-)^{\vee\vee}$. Such a pair $(\mathcal{C}, \omega)$ is called a pivotal category.

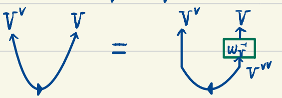
Prop 2.12. Every pivotal category is left rigid, whose left and right duals coincide.

Pf. We can define ${}_V ev$ and ${}_V coev$, for $\forall V \in \mathcal{C}$, as follows,

$${}_V ev : V \otimes V^{\vee} \xrightarrow{\omega_V \otimes id_{V^\vee}} V^{\vee\vee} \otimes V^{\vee} \xrightarrow{ev_{V^\vee}} I.$$

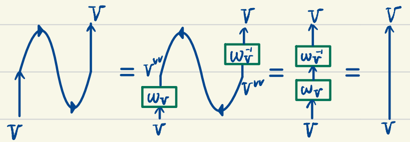


$${}_V coev : I \xrightarrow{coev_{V^\vee}} V^{\vee} \otimes V^{\vee\vee} \xrightarrow{id_{V^\vee} \otimes \omega_V^{-1}} V^{\vee} \otimes V.$$



One can show that:

$({}^{\vee}V := V^{\vee}, {}_V ev, {}_V coev)$ gives a left dual, for example:



$\square$

Example.

1). $Vect^{f}_{\Bbbk}$ is pivotal with $\omega^{can}_V : v \longmapsto (f \in V^* \mapsto f(v))$.

2). $\mathcal{J}$, $f\mathcal{J}$, $C\text{-}\mathcal{J}$, $C\text{-}f\mathcal{J}$ are all pivotal with the identy natural morphism as a pivotal structure.

Rem.

Notice that for a pivotal category, the pivotal structure might not be unique!

Def 2.13. (Pivotal Hopf algebra)

A pivotal Hopf algebra is a Hopf algebra H with an element $g \in H$, such that

- g is grouplike, i.e., $\Delta g = g \otimes g$,
- $S^2(h) = g \cdot h \cdot g^{-1}$ $\forall h \in H$. (g is almost central)

Prop 2.14.

For a Hopf algebra H, $Rep^{\dagger}_{\mathbb{K}}(H)$ is pivotal if H is pivotal.

Pf. $\forall V \in Rep^{\dagger}_{\mathbb{K}}(H)$, define $\omega_V := \omega_V^{can}\circ(g.)$, where g is a pivotal element in H and $\omega_V^{can}: v \mapsto (f\in V^* \longmapsto f(v))$.

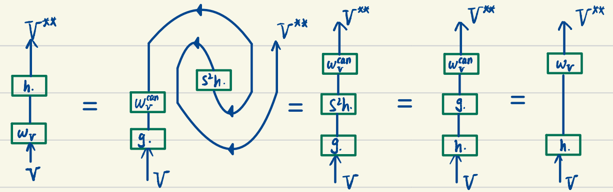
One can check that:

1) ω_V is H-equivariant. $\Longleftarrow$ $S^2 h = g h g^{-1}$ $\forall h\in H$.

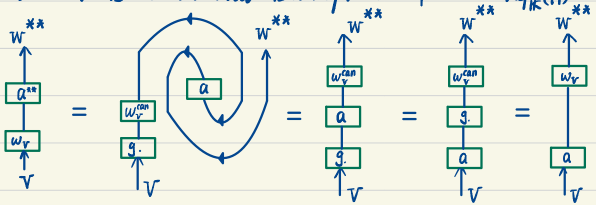
$\forall v\in V, f\in V^*$: $\omega_V(h.v)(f) = \omega_V^{can}(gh.v)(f) = f(S^2(h)g.v)$

$= S(h).f(g.v) = \omega_V^{can}(g.v)(S(h).f) = h.\omega_V(v)(f)$

Hence, $\omega_V(h.v) = h.\omega_V(v)$.



2) ω_V is a natural isomorphism from $Id_{Rep^{\dagger}_{\mathbb{K}}(H)}$ to $(-)^{\vee\vee}$.



$\forall\, a\in \ell(V,W)$.

$\omega_V^{-1} = (g^{-1}.)\circ(\omega_V^{can})^{-1} = (S(g).)\circ(\omega_V^{can})^{-1}$

3) ω_V is monoidal. $\Longleftarrow$ g is grouplike.

$\omega_{V\otimes W} = \omega^{can}_{V\otimes W}\cdot(g.) = \omega^{can}_{V\otimes W}\cdot(g.\otimes g.)$

$= (\omega_V^{can}\cdot(g.))\otimes(\omega_V^{can}\cdot(g.)) = \omega_V\otimes\omega_W$.

$\square$

Rem.

In cases where H is finite dimensional, the converse statement also holds. We can define a pivot in H by

$g := (\omega_H^{can})^{-1}\circ\omega_H(\eta(1)) \in H$.

One can show that

1) g is grouplike,

$\Delta g = \Delta\circ(\omega_H^{can})^{-1}\circ\omega_H(\eta(1)) = (\omega^{can}_{H\otimes H})^{-1}\circ\omega_{H\otimes H}(\Delta\eta(1)) = (\omega^{can}_{H\otimes H})^{-1}\circ\omega_{H\otimes H}(\eta(1)\otimes\eta(1))$

$=((\omega_H^{can})^{-1}\omega_H(\eta(1)))\otimes((\omega_H^{can})^{-1}\circ\omega_H(\eta(1))) = g\otimes g$

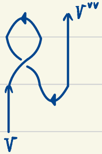
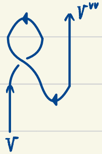
2) $S^2(h)g = gh$, $\forall h\in H$.

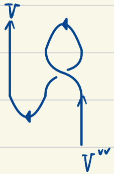
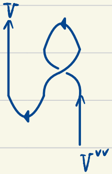
$S^2(h)g = (S^2(h).)\circ(\omega_H^{can})^{-1}\circ\omega_H(\eta(1)) = (\omega_H^{can})^{-1}\circ(h.)\circ\omega_H(\eta(1)) = (\omega_H^{can})^{-1}\circ\omega_H(h)$

$= (\omega_H^{can})^{-1}\circ\omega_H\circ R_h(\eta(1)) = R_h\circ(\omega_H^{can})^{-1}\circ\omega_H(\eta(1)) = gh$.

Prop 2.15. Every right rigid braided monoidal category $(\mathcal{C}, c)$ carries two natural isomorphisms $u, v^{-1}: \mathrm{Id}_{\mathcal{C}} \xrightarrow{\simeq} (-)^{\vee\vee}$. If c are symmetric, $u = v^{-1}$ gives a pivotal structure.

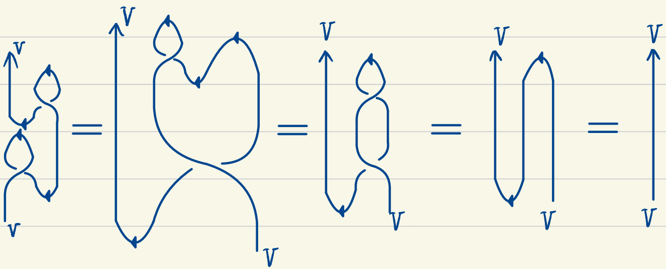
pf. Define

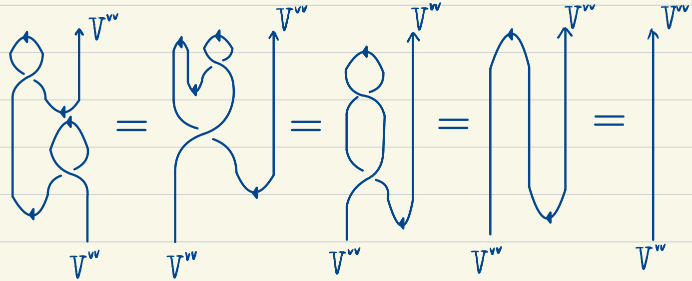
$u_V :=$ 
$v_V^{-1} :=$ 

$u_V^{-1} :=$ 
$v_V :=$ 

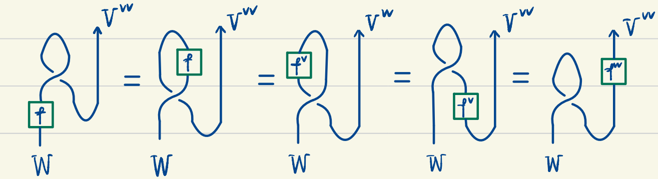
One can show that :

1). $u_V^{-1} \circ u_V = id_V = v_V \circ v_V^{-1}$, $u_V \circ u_V^{-1} = id_{V^{\vee\vee}} = v_V^{-1} \circ v_V$

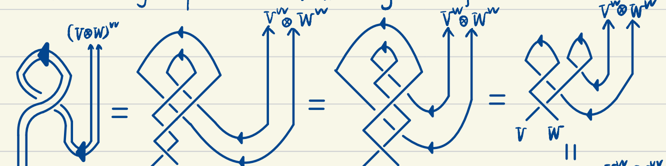


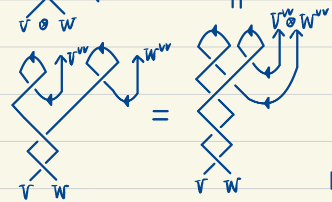


2). u and v^{-1} are natural. $\forall f \in \mathcal{C}(W, V)$,



Moreover, by definition, $u_{V\otimes W}$ is given as follows.



$J_{V,W} \circ (u_V \otimes u_W) \circ c_{W,V}^{-1} \circ c_{V,W}^{-1}$ = 

□

Prop 2.16. If $(H, \mathcal{R})$ is a quasi-triangular Hopf algebra. Then the natural isomorphisms $u, v^{-1} : Id_{Rep^f_{\mathbb{K}}(H)} \to (-)^{**}$ defined as above are induced by the action of Drinfel'd elements $u := \sum S\mathcal{R}^{(2)}\mathcal{R}^{(1)}$, $v^{-1} := \sum S^2\mathcal{R}^{(1)}\mathcal{R}^{(2)}$, i.e. $\forall V \in Rep^f_{\mathbb{K}}(H)$, $u_V = \omega_V^{can} \circ (u.)$ and $v_V^{-1} = \omega_V^{can} \circ (v^{-1}.)$.

Pf. $\forall e \in V, f \in V^*$,

$$u_V(e)(f) = \mathcal{R}^{(2)}.f(\mathcal{R}^{(1)}.e) = f(S\mathcal{R}^{(2)}\mathcal{R}^{(1)}.e)$$
$$= \omega_V^{can}(u.e)(f).$$

$$v_V^{-1}(e)(f) = S\mathcal{R}^{(1)}.f(\mathcal{R}^{(2)}.e) = f(S^2\mathcal{R}^{(1)}\mathcal{R}^{(2)}.e)$$
$$= \omega_V^{can}(v^{-1}.e)(f).$$

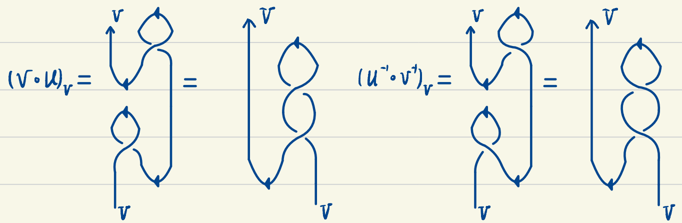
$\square$

Question: How to modify u and v^{-1} into a pivotal structure?

Idea:
$$\begin{cases} S^2h\,u = uh,\ vS^2h = hv & \forall h \in H \\ \Delta u = (u \otimes u)\mathcal{R}^{-1}\mathcal{R}_{21}^{-1}, & \Delta v^{-1} = (v^{-1} \otimes v^{-1})\mathcal{R}_{21}\mathcal{R} \end{cases}$$

A candidate for a pivot is "a balance of u and v^{-1}", i.e., an element $g = u\theta^{-1} = v^{-1}\theta$, such that θ is central and $\Delta\theta = (\theta \otimes \theta)\mathcal{R}^{-1}\mathcal{R}_{21}^{-1}$. $\Rightarrow$ ribbon element.

Equivalently, for a right rigid braided monoidal category $(\mathcal{C}, c)$, to make it pivotal, we need to balance u and v^{-1}. Notice that,



the natural isomorphism $\theta : Id_{\mathcal{C}} \to Id_{\mathcal{C}}$ to balance u and v^{-1} should be a square root of $v \circ u$, which is a 1-twist.

$\Longrightarrow$ ribbon category.

§3. Ribbon categories and operator invariants

Def 3.1. Let $(\mathcal{C}, \otimes, I, a, l, r, c)$ be a braided monoidal category.

1). A *twist* is a natural isomorphism $\theta : Id_{\mathcal{C}} \to Id_{\mathcal{C}}$, such that

- $\theta_{V\otimes W} = (\theta_V \otimes \theta_W) \circ c_{W,V}^{-1} \circ c_{V,W}^{-1}$.

$(\mathcal{C}, c)$ with a twist θ is called a *balance monoidal category*.

2). If $(\mathcal{C}, c, \theta)$ is also right rigid, and $\forall V \in \mathcal{C}$,

- $(\theta_V)^\vee = \theta_{V^\vee}$,

θ is called a *ribbon structure*, and $(\mathcal{C}, c, \theta)$ is called a *ribbon category*.

Prop 3.2. Let $(\mathcal{C}, c, \theta)$ be a ribbon category. Then,

1). $\theta_I = id_I$

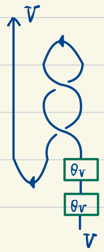
2). $\theta^2 = v \circ u$, or equivalently, $u \circ \theta^{-1} = v^{-1} \circ \theta$.

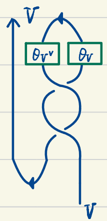
Pf. 1). $\theta_{I\otimes I} = \theta_I \otimes \theta_I$. Hence,

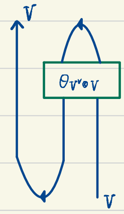
$$\theta_I \circ l_I = l_I \circ \theta_{I\otimes I} = l_I \circ (\theta_I \otimes \theta_I) = l_I \circ (id_I \otimes \theta_I) \circ (\theta_I \otimes id_I) = \theta_I \circ l_I \circ (\theta_I \otimes id_I)$$

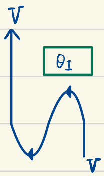
$$= \theta_I \circ r_I \circ (\theta_I \otimes id_I) = \theta_I^2 \circ r_I = \theta_I^2 \circ l_I$$

Then, $\theta_I = \theta_I^{-1} \circ \theta_I^2 \circ l_I \circ l_I^{-1} = \theta_I^{-1} \circ \theta_I \circ l_I \circ l_I^{-1} = id_I$.

2). $(v^{-1} \circ u^{-1} \circ \theta^2)_V =$ 

$=$ 

$=$ 

$=$ 

$=$ 

□

Prop 3.3. A ribbon category is pivotal with

$$\omega := u \circ \theta^{-1} = v^{-1} \circ \theta.$$

Pf. Only need to show that ω is monoidal. $\forall V, W \in \mathcal{C}$,

$$\omega_{V\otimes W} = u_{V\otimes W} \circ \theta^{-1}_{V\otimes W}$$

$$= (u_V \otimes u_W) \circ c^{-1}_{W,V} \circ c^{-1}_{V,W} \circ c_{V,W} \circ c_{W,V} \circ (\theta_V^{-1} \otimes \theta_W^{-1})$$

$$= \omega_V \otimes \omega_W.$$

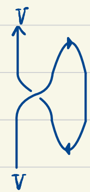
□

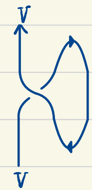
Coro 3.4. $(\mathcal{C}, c, \theta)$ is a ribbon category. Take the left dual structure of $\mathcal{C}$ induced by the pivotal structure ω defined as above, i.e., $\forall V \in \mathcal{C}$,

$$({}^{\vee}V = V^{\vee},\ {}_{V}ev = ev_{V^{\vee}} \circ (\omega_V \otimes id_{V^{\vee}}),\ {}_{V}coev = (id_{V^{\vee}} \otimes \omega_V^{-1}) \circ coev_{V^{\vee}})$$

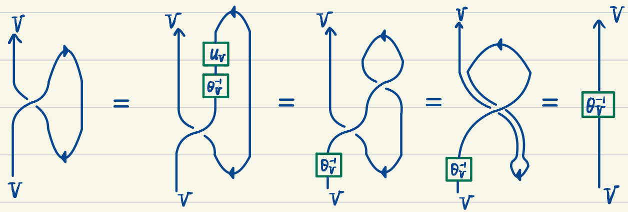
Then $\forall V \in \mathcal{C}$,

- $\theta_V = r_V \circ (id_V \otimes {}_{V}ev) \circ a_{V,V,V^{\vee}} \circ (c^{-1}_{V,V} \otimes id_{V^{\vee}}) \circ a^{-1}_{V,V,V^{\vee}} \circ (id_V \otimes coev_V) \circ r_V^{-1}$,
- $\theta_V^{-1} = r_V \circ (id_V \otimes {}_{V}ev) \circ a_{V,V,V^{\vee}} \circ (c_{V,V} \otimes id_{V^{\vee}}) \circ a^{-1}_{V,V,V^{\vee}} \circ (id_V \otimes coev_V) \circ r_V^{-1}$,

i.e., $\theta_V^{-1} =$ 

$\theta_V =$ 

pf:

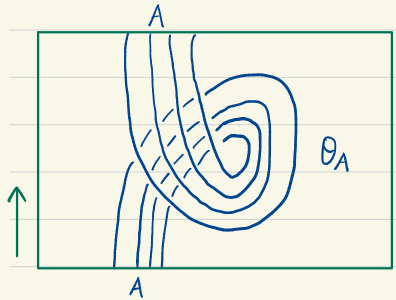


$\square$

Example.

1). Any symmetric monoidal category is ribbon with the pivotal structure $u = v^{-1}$ and the ribbon structure $\theta = id$.

2). $\mathcal{T}$, $C\text{-}\mathcal{T}$, $f\mathcal{T}$, $C\text{-}f\mathcal{T}$ is ribbon, with the ribbon structures



- Orientation on each strand is determined by '$\pm$'-color of boundary points
- All crossings are '$\times$'.

Notice that

- θ is trivial for $\mathcal{T}$ and $C\text{-}\mathcal{T}$ by Riedemeister move I.
- θ^{-1} is by replacing all crossing in θ from '$\times$' to '$\times$'.
- For $f\mathcal{T}$, $C\text{-}f\mathcal{T}$, θ is not trivial and $\theta^{-1} \circ \theta = id = \theta \circ \theta^{-1}$ is equivalent to fRI.

Recall. A ribbon Hopf algebra is a quasi-triangular Hopf algebra $(H, \mathcal{R})$ with an invertible central element $\theta \in H$, such that

- $\theta^2 = u\,S(u) = u v^{-1}$,
- $\Delta\theta = (\theta \otimes \theta)\mathcal{R}^{-1}\mathcal{R}_{21}^{-1}$.

θ is called a ribbon element.

Rem.
The definition above is equivalent to that in textbooks, since we have

$$\theta = (\varepsilon \otimes id)\Delta\theta = (\varepsilon \otimes id)(\theta\otimes\theta)\mathcal{R}^{-1}\mathcal{R}_{21}^{-1} = \varepsilon(\theta)\theta$$

$$\Rightarrow \quad \varepsilon(\theta) = \varepsilon(\theta)\theta\theta^{-1} = 1$$

$$S\theta\cdot\theta = m(S\otimes id)(\theta\otimes\theta) = m(S\otimes id)(\Delta\theta\,\mathcal{R}_{21}\mathcal{R})$$
$$= S\mathcal{R}_1^{(1)} S\mathcal{R}_2^{(2)} S\theta_{(1)}\theta_{(2)} \mathcal{R}_2^{(1)} \mathcal{R}_1^{(2)}$$
$$= \varepsilon(\theta)\, u\, S^{-1}\mathcal{R}_1^{(1)}\mathcal{R}_1^{(2)} = \varepsilon(\theta)\, u\, S(u) = u\,S(u) = \theta^2$$

$$\Rightarrow \quad S\theta = \theta.$$

Prop 3.5. If $(H, \mathcal{R}, \theta)$ is a ribbon Hopf algebra, then $\mathrm{Rep}_{\mathbb{K}}^{f}(H)$ is a ribbon category with a ribbon structure defined the action of θ.

pf.
1). $(\theta.)$ is H-equivariant. $\Leftarrow$ θ is central.
2). $(\theta.)$ gives a natural isomorphism. $\Leftarrow$ $\theta\in H$ is invertible.
3). $(\theta.)$ is a twist. $\Leftarrow$ $\Delta\theta = (\theta\otimes\theta)\mathcal{R}^{-1}\mathcal{R}_{21}^{-1}$.
4). $(\theta.)^\vee = (\theta.)$ on V^* $\Leftarrow$ $S\theta = \theta$.

$\square$

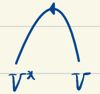
Coro 3.6. For a ribbon Hopf algebra $(H, \mathcal{R}, \theta)$, the pivot $g \in H$ corresponding the pivotal structure ω on $\mathrm{Rep}_{\mathbb{K}}^{f}(H)$ is given by $g = u\theta^{-1} = v^{-1}\theta$.

Rem.
The converse of Prop 3.5 is also true if H is finite dimensional.

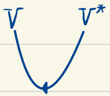
Thm 3.7.

Given any ribbon category $(\mathcal{C}, c, \theta)$ and $V \in \mathcal{C}$, there exists a braided monoidal functor $F_V : f\mathcal{T} \longrightarrow \mathcal{C}$, such that. $F_V(+) = V$, $F_V(-) = V^*$ and

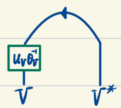
- $F_V(\overleftarrow{n}) = ev_V : V^* \otimes V \to \mathbb{k}$,



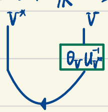
- $F_V(\overleftarrow{u}) = coev_V : \mathbb{k} \longrightarrow V \otimes V^*$,



- $F_V(\vec{n}) = {}_V ev : V \otimes V^* \longrightarrow \mathbb{k}$



- $F_V(\vec{u}) = {}_V coev : \mathbb{k} \longrightarrow V^* \otimes V$



F_V is unique up to monoidal natural isomorphism and composed of associative constraints and unit constraints.

pf. By strictness theorem, we can assume $\mathcal{C}$ and F_V are strict. Then $F_V(\phi) = \mathbb{k}$, $F_V(\epsilon_1, \cdots, \epsilon_n) = F_V(\epsilon_1) \otimes \cdots \otimes F_V(\epsilon_n)$ $\epsilon_i = \pm$. Hence, on objects of $f\mathcal{T}$, F_V is uniquely determined by $F_V(+) = V$ and $F_V(-) = V^*$.

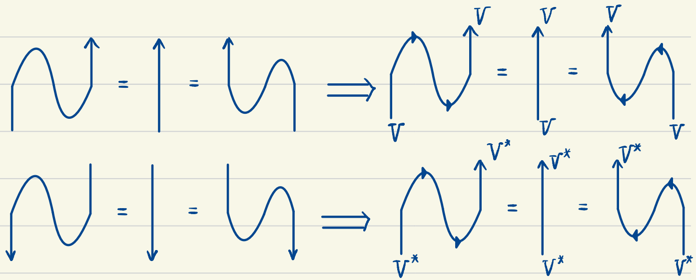
On morphisms, F_V is uniquely determined by the image of $\vec{n}, \vec{u}, \overleftarrow{n}, \overleftarrow{u}$ and $\nwarrow\!\!\!\!\nearrow$, $\nwarrow\!\!\!\!\nearrow$. Since F_V is braided,

$F_V(\times) = c_{V,V}$, $F_V(\times) = c^{-1}_{V,V}$.

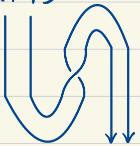
To finish the proof, we only need to check F_V is invariant for framed Turaev moves:

(fT0) Trivial since F_V is a monoidal functor,

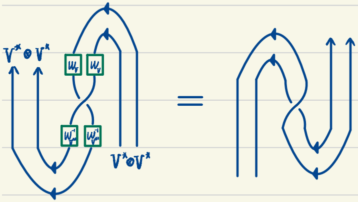
(fT1) (fT2).



(fT3)

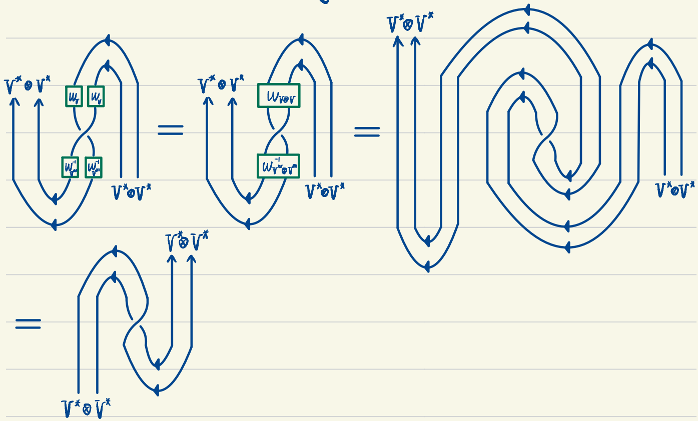


$\Rightarrow$

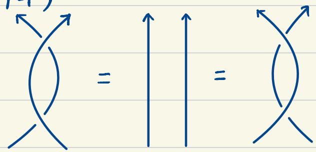


, Here $w = u \circ \theta^{-1}$ is the pivotal structure.

This holds because the folling calculation,

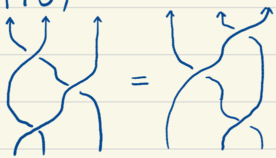


(fT4)



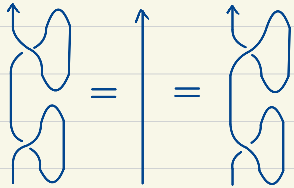
Trivial by the braiding structure.

(fT5)



Trivial by the braiding structure.

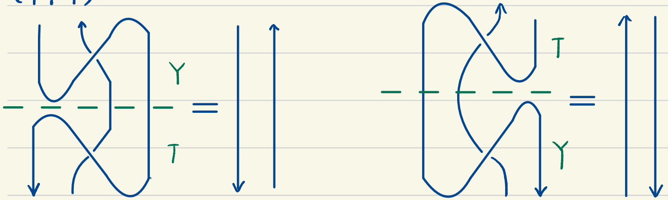
(fT6)



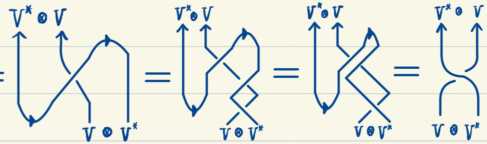
$\Rightarrow \quad \theta_V^{-1} \circ \theta_V = id_V = \theta_V \circ \theta_V^{-1}$

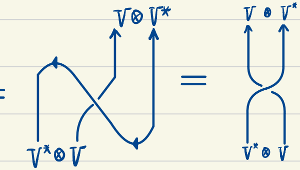
Notice that this is the reason we need ribbon structure. We need a nontrivial θ to encode the ribbon structure of fT.

(fT7)



Notice that

$F_V(Y) =$  $= c^{-1}_{V\otimes V^*}$

$F_V(T) =$  $= c_{V^*\otimes V}$.

$\square$

$$\begin{tikzcd} f\mathcal{T} \arrow[r, "F"] \arrow[dr, "Q^{f}_{V,V^*,R,\alpha,\beta}"'] & \mathrm{Rep}^{f}_{\mathbb{k}}(H) \arrow[d, "\mathrm{Forget}"] \\ & \mathrm{Vect}^{f}_{\mathbb{k}} \end{tikzcd}$$

Coro 3.8. Let $(H, \mathcal{R}, \theta)$ be a ribbon Hopf algebra.

Take a collection C of finite dimensional $\mathbb{k}$-modules of H.

Then up to monoidal natural isomorphism and composed of associative constraints and unit constraints, there exists a unique braided monoidal functor $F: C\text{-}f\mathcal{T} \longrightarrow \mathrm{Rep}^{f}_{\mathbb{k}}(H)$, such that

$F(V_i, +) = V_i$, $F(V_i, -) = V_i^*$, $\forall V_i \in C$ and

$F(\overleftarrow{\cup}_{V_i}) = \mathrm{coev}_{V_i}$, $F(\overleftarrow{\cap}_{V_i}) = \mathrm{ev}_{V_i}$,

$F(\overrightarrow{\cup}_{V_i}) = {}_{V_i}\mathrm{coev}$, $F(\overrightarrow{\cap}_{V_i}) = {}_{V_i}\mathrm{ev}$.

where $\mathrm{coev}_{V_i}, \mathrm{ev}_{V_i}$, are defined in Prop 2.8

${}_{V_i}\mathrm{coev}, {}_{V_i}\mathrm{ev}$ are defined in Prop 2.12 by the pivotal structure defined in Prop 3.3. and Prop 3.5.

Especially, when $C = \{V\}$, and denote by Forget the forgetful functor

$$\mathrm{Forget}: \mathrm{Rep}^{f}_{\mathbb{k}}(H) \longrightarrow \mathrm{Vect}^{f}_{\mathbb{k}}$$

we get back an operator invariant of framed tangles.

- $R = c_{V,V} = \tau \circ (\mathcal{R}^{(1)}. \otimes \mathcal{R}^{(2)}.)$
- $\alpha = (u^{-1}\theta.) \circ (w^{can}_{V})^{-1} = w_V^{-1}: V^{**} \to V$, $\beta = (w^{can}_V)^{-1}: V^{**} \to V$
- In particular, $\mu = \beta \circ \alpha^{-1} = (u\theta^{-1}). : V \to V$ is the action by the pivot of $(H, \mathcal{R}, \theta)$.

Example.

Consider $U_q(\mathfrak{sl}_2)$, generated by E, F, K, K^{-1} with relations

$$\begin{cases} EF - FE = \dfrac{K - K^{-1}}{q^{1/2} - q^{-1/2}} \\ K \cdot K^{-1} = K^{-1} \cdot K = 1 \\ KE = qEK \\ KF = q^{-1} FK \end{cases}$$

Imaging $q = e^{\hbar}$, $K = q^{H/2} = e^{\hbar H/2}$,

$$HE = E(H+2), \quad HF = F(H-2), \quad [E,F] = [H]_q$$

Denote $[n]_q = \dfrac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$.

$$\Delta K^{\pm 1} = K^{\pm 1} \otimes K^{\pm 1}, \quad SK^{\pm 1} = K^{\mp 1}, \quad \varepsilon(K^{\pm 1}) = 1.$$

K is grouplike, in fact the pivot.

$$\Delta E = E \otimes K + 1 \otimes E, \quad SE = -EK^{-1}, \quad \varepsilon(E) = 0.$$

$$\Delta F = F \otimes 1 + K^{-1} \otimes F, \quad SF = -KF, \quad \varepsilon(F) = 0.$$

$$\mathcal{R} = q^{H \otimes H/4} \sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}}{[n]_q!} \left((q^{1/2} - q^{-1/2}) E \otimes F \right)^n$$

$$\theta = q^{-H^2/4} \sum_{n=0}^{\infty} q^{n(3n+1)/4} \frac{(q^{-1/2} - q^{1/2})^n}{[n]_q!} F^n K^{-n-1} E^n.$$

Consider a 2-dim'l. representation V with

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad K = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}$$

$$R = \tau \circ (\mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}) = \begin{pmatrix} q^{1/4} & 0 & 0 & 0 \\ 0 & 0 & q^{-1/4} & 0 \\ 0 & q^{-1/4} & q^{1/4} - q^{-3/4} & 0 \\ 0 & 0 & 0 & q^{1/4} \end{pmatrix}$$

$$K = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}$$

The corresponding operator invariants $Q = Q^f_{V, V^*, R, \alpha, \beta}$ satisfies skein relation,

$$q^{1/4} Q(\quad) - q^{-1/4} Q(\quad) = (q^{1/2} - q^{-1/2}) Q(\quad)$$

In fact,

$$Q(L) = (-1)^{\#L + f(L)} \langle L \rangle \big|_{A = q^{1/4}}$$

$\#L = \#$ components of L

$f(L) =$ sum of twists of each components.