Tangles and Operator Invariants

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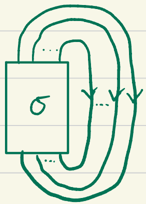
§1. Motivation

To find a systematical way to generalize the link invariant $L \mapsto P_{R,\mu}(L) \in \mathbb{k}$.

Recall: To define $P_{R,\mu}(L)$, we need,

Step 1: Find a braid σ, s.t. $\hat{\sigma} = L$.



$L = \hat{\sigma}$

Step 2: Find a representation for the braid group via 'elementary pieces', generators of a braid and 'braid relations', YB equation.

- a free finite rank $\mathbb{k}$-mod V
- a morphism $\rho: \mathcal{B}_n \to \mathrm{End}(V^{\otimes n})$ generated by

$$\rho(\times) = R \in \mathrm{End}(V^{\otimes 2})$$

$$\rho(\times) = R^{-1} \in \mathrm{End}(V^{\otimes 2})$$

via operations $\otimes$, $\circ$. s.t.

$$\rho(\sigma_1 \otimes \sigma_2) = \rho(\sigma_1) \otimes \rho(\sigma_2)$$

$$\rho(\sigma_1 \circ \sigma_2) = \rho(\sigma_1) \circ \rho(\sigma_2)$$

and R is an R-matrix.

Step 3: Find an isomorphism $\mu \in \mathrm{End}(V)$, satisfies

- $R \circ (\mu \otimes \mu) = (\mu \otimes \mu) \circ R$.
- $\mathrm{Tr}_2(R \circ (id \otimes \mu)) = id. \Leftrightarrow \mathrm{Tr}_2(R \circ (\mu \otimes \mu)) = \mu$.

$$P_{R,\mu}(L) := \mathrm{Tr}(\rho(\sigma) \circ \mu^{\otimes n}) \in \mathbb{k}.$$

In other words, we regard $\bigsqcup_n B_n$ as a strict braided monoidal groupoid $(\mathcal{B}, \otimes, I)$:

- Objects: $[n]$ = a total-ordered set of n points, $n \in \mathbb{N} \cup \{0\}$. $I = [0] = \phi$.
- Morphisms: $\mathcal{B}([n],[m]) = \begin{cases} \phi & \text{if } n \neq m \\ B_n & \text{if } n = m. \end{cases}$

and define P as a monoidal functor

$$P: (\mathcal{B}, \otimes, I) \to (\Bbbk\text{-}\overset{f.f.}{\text{mod}}, \otimes_{\Bbbk}, \Bbbk).$$
$$[1] \mapsto V$$

Hence, to generalize $P_{R,\mu}$, we need to:

i). generalize $(\mathcal{B}, \otimes, I)$ to a 'bigger', 'nice' category to include (oriented) links directly.

ii). find a 'nice' functor with values in $\Bbbk\text{-}\overset{f.f.}{\text{mod}}$.

iii). recover μ.

§2. Tangles and tangle diagrams.

Def 1. (Tangle) An (m,n)-**tangle** T consists of a disjoint union of copies of $I = [0,1]$ and S^1, embedded into $\mathbb{R}^2 \times I$ such that:

1). $\partial T = \{(a_i, 0, 0) \mid i = 1, \cdots, m\} =: S(T)$
$\sqcup \{(b_i, 0, 1) \mid i = 1, \cdots, n\} =: t(T)$

$a_1 < \cdots < a_m$, $b_1 < \cdots < b_n$

2). $T \cap \partial(\mathbb{R}^2 \times I) = \partial T$.

Each I or S^1 is called a **component** of T.

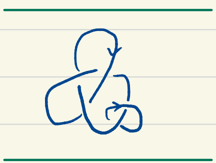
T is **oriented** if each component is oriented.

In oriented cases, we color each point in $s(T)$ and $t(T)$ by '+' and '−' in the following way:

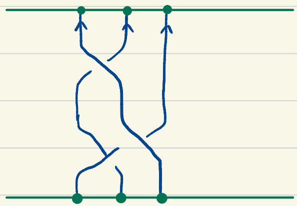
- for a point in $s(T)$,
  - if it is the start point of a component it will be colored by '+';
  - if it is the end point of a component it will be colored by '−';
- for a point in $t(T)$,
  - if it is the start point of a component it will be colored by '−';
  - if it is the end point of a component it will be colored by '+';

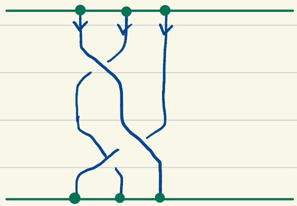
<u>Example</u>.

1). Any knots or links are $(0,0)$-tangles.

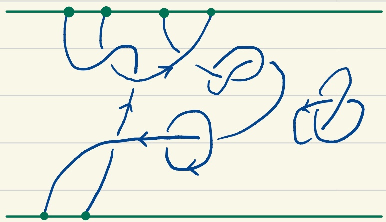


2). Any n-strand braid is an (n,n)-tangle.





3).

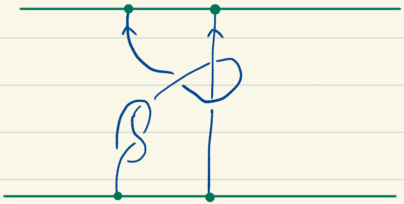


$\mathbb{R}\times\{0\}\times\{0\}$

$\mathbb{R}\times\{0\}\times\{1\}$

A $(4,2)$-tangle with $(3+2)$ components.

4).



a string link

sting link: (n,n)-tangle with no closed components and i-strand from i-th input to i-th output.

pure braid

if it is also a braid

long knot

if only one knot.

Def 2. (Tangle equivalence).

- Two (oriented) tangles T and T' are said to be isotopy equivalent, if there exists a boundary-preserving (and orientation preservating) isotopy of $\mathbb{R}^2 \times I$ taking T to T'. Thus, a family of homeomorphisms,

$$h_t : \mathbb{R}^2 \times I \to \mathbb{R}^2 \times I, \quad t \in [0,1].$$

such that:

1). $h_t|_{\mathbb{R}^2 \times \{0,1\}} = id$.

2). $h_0 = id$.

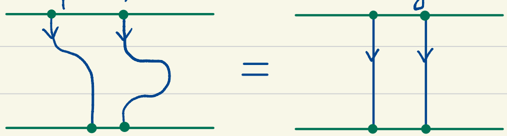
3). $h_1(T) = T'$.

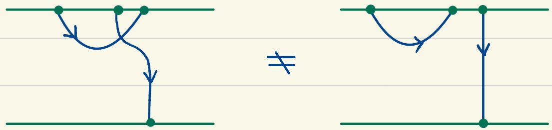
4). $(x,t) \mapsto (h_t(x), t)$ defines a homeomorphism from $\mathbb{R}^2 \times I^2$ to itself.

- Two tangles are said to be equivalent, if they are isotopy equivalent up to horizontal rescaling.

$$(x, y, z) \mapsto (f(x,z), y, z),$$

where $f(x,z)$ is smooth and strictly increasing.

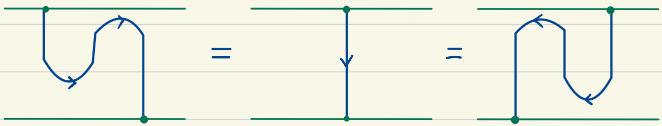




Rem:

1). Up to equivalence, $\underset{\substack{\shortparallel \\ \partial T}}{s(T) \sqcup t(T)}$ is encoded by two total ordered sets of '$\pm$'-colored points.

2). Unlike the braids cases, tangles allow critical parts.

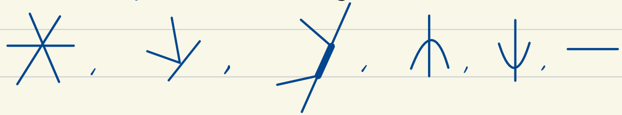


3). We only consider tame tangles.

Tameness $\Leftrightarrow$ equivalent to a polygon tangle.

Equivalence between polygon tangles is given by a finite sequence of $\Delta^{\pm}$-moves.

4). By the projection $\mathbb{R}^2 \times I \to \mathbb{R} \times \{0\} \times I$, we have the corresponding tangle diagrams. We also require such diagrams to be regular.

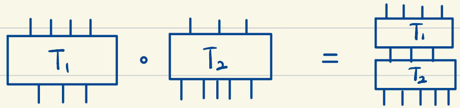


Def 3. (Tangle operations)

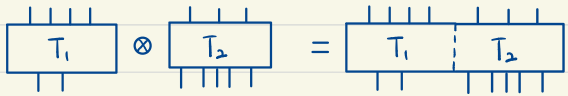
1). For two tangles T_1, T_2, if $s(T_1) \overset{\downarrow}{=} t(T_2)$, (as ordered set of '$\pm$'-colored points)

we can define their composition product $T_1 \circ T_2$,

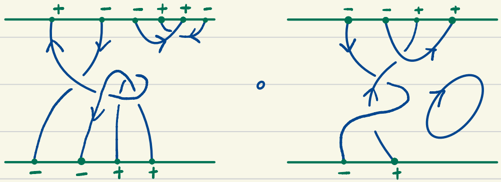
as



2). For any two tangles T_1 and T_2, we can also define their tensor product $T_1 \otimes T_2$,



Example.



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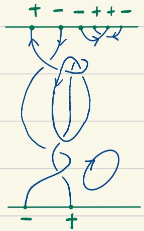
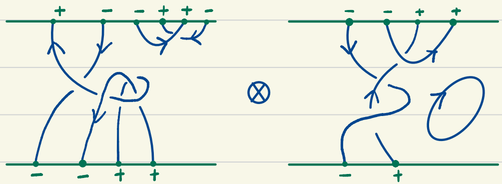


$$\begin{cases} s(T_1 \circ T_2) = s(T_2) \\ t(T_1 \circ T_2) = t(T_1) \end{cases}$$



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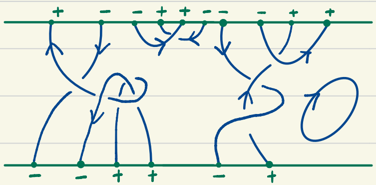


$$\begin{cases} s(T_1 \otimes T_2) = s(T_1) \otimes s(T_2) \\ t(T_1 \otimes T_2) = t(T_1) \otimes t(T_2) \end{cases}$$

Here, these two operations give rise to operations on the equivalent classes of tangles. Similar as cases of braids, (oriented) tangles compose a strict monoidal category $(\mathcal{T}, \otimes, I = \phi)$:

Obj.: total ordered set of '$\pm$'-colored points.

Mor.: equivalent classes of tangles, $[T] \in \mathcal{T}(A, B)$ if $s(T) = A$, $t(T) = B$.

I: identity object given by the empty set.

Rem.

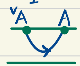
1). $\mathcal{T}$ is not a groupoid any more.

2). $\mathcal{T}$ contains more structures over its monoidal structure:

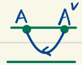
- rigidity: each object has a left/right dual.

  Example. $A = \{+\}$, then $A^{\vee} = {}^{\vee}A = \{-\}$ and the corresponding left/right (co)evaluation maps are,

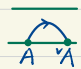
$_A\mathrm{coev}: I \to {}^{\vee}A \otimes A$



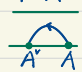
$\mathrm{coev}_A: I \to A \otimes A^{\vee}$



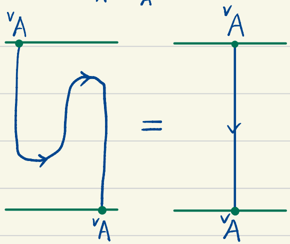
$_A\mathrm{ev}: A \otimes {}^{\vee}A \to I$



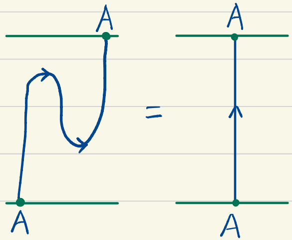
$\mathrm{ev}_A: A^{\vee} \otimes A \to I$



${}^\vee A$ with ${}_A ev$, ${}_A coev$ is the left dual of A, because



$\Leftrightarrow (id_{{}^\vee A} \otimes {}_A ev) \circ ({}_A coev \otimes id_{{}^\vee A}) = id_{{}^\vee A}$



$\Leftrightarrow ({}_A ev \otimes id_A) \circ (id_A \otimes {}_A coev) = id_A$.

Similar for $A^\vee$, ev_A, $coev_A$.

For a strict rigid monoidal category $(\mathcal{C}, \otimes, I)$, there is canonical isomorphisms:

$$\mathcal{C}({}^\vee B, A) \xleftarrow{\cong} \mathcal{C}(I, A \otimes B) \xrightarrow{\cong} \mathcal{C}(A^\vee, B)$$

$$(id_A \otimes {}_B ev) \circ (f \otimes id_{{}^\vee B}) \longleftarrow f \longmapsto (ev_A \otimes id_B) \circ (id_{A^\vee} \otimes f)$$

$$\mathcal{C}(B, {}^\vee A) \xleftarrow{\cong} \mathcal{C}(A \otimes B, I) \xrightarrow{\cong} \mathcal{C}(A, B^\vee)$$

$$(id_{{}^\vee A} \otimes f) \circ ({}_A coev \otimes id_B) \longleftarrow f \longmapsto (f \otimes id_{B^\vee}) \circ (id_A \otimes coev_B)$$

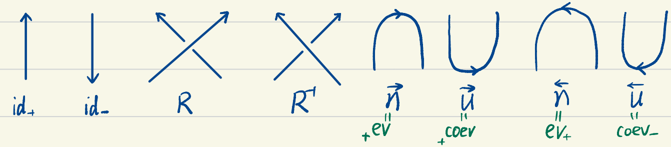
However, the (co)evaluation maps are not unique!

- $(\mathcal{T}, \otimes, I)$ is also a braided monoidal category, but not a symmetric monoidal category.
- $(\mathcal{T}, \otimes, I)$ is in fact a ribbon category, which will be discussed in next two talks.

Def 4. (Elementary and simple tangles diagrams.)

1). Unoriented elementary tangle diagrams.

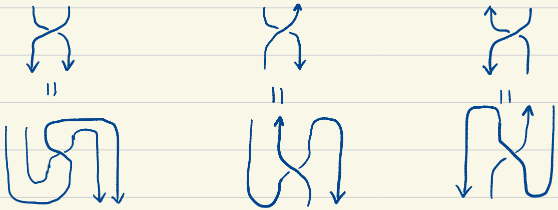
2). Oriented elementary tangle diagrams.



3). A simple tangle diagram is a tensor product of (maybe none) elementary ones.

Especially, an identity tangle is a simple tangle with only vertical lines, i.e., in the form of $id_+^{\otimes i_1} \otimes id_-^{\otimes i_2} \otimes \cdots$.

<u>Rem</u>. Other types of crossings are equivalent to tangles generated by elementary ones. For example,

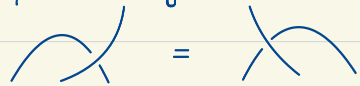


Similar as cases of links, we have

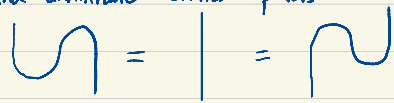
$$\{\text{tangles}\}\big/\text{isotopy equivalence} \longleftrightarrow \frac{\{\text{tangle diagrams}\}\big/\text{isotopy of } \mathbb{R}\times I}{\text{Riedemeister moves}}$$

and isotopy of $\mathbb{R}\times I$ are:

i). rotation of a crossing



ii). create and annihilate critical points



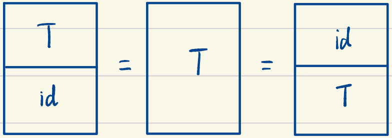
iii). Other isotopies preserving crossings and critical points.

<u>Lem 5.</u> Up to isotopies of $\mathbb{R}\times I$, each (oriented) tangle diagram can be sliced into simple tangle diagrams via horizontal lines, i.e., it can be obtained by composition of finitely many tensor products of (oriented) elementary tangle diagram. We shall call it a sliced tangle diagram.

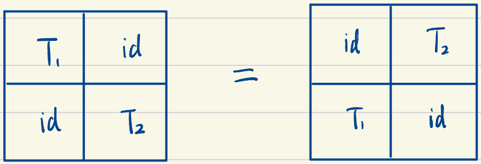
Thm 6.

1). Two unoriented sliced tangle diagrams are equivalent iff they are related by a finite sequence of unoriented Turaev moves,

(T_0^u)

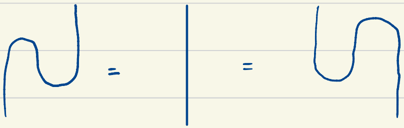


$T \circ id = T = id \circ T.$

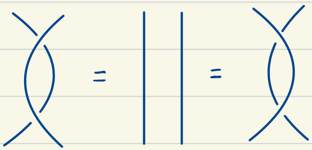


$(T_1 \otimes id) \circ (id \otimes T_2) = (id \otimes T_2) \circ (T_1 \otimes id)$

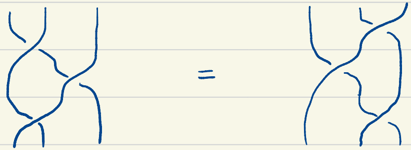
(T_1^u)



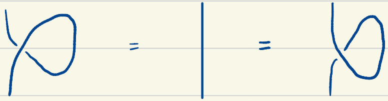
(T_2^u)



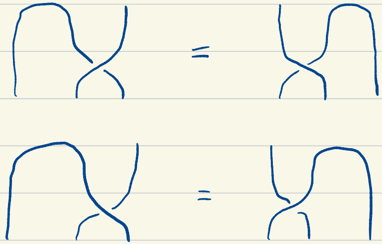
(T_3^u)



(T_4^u)



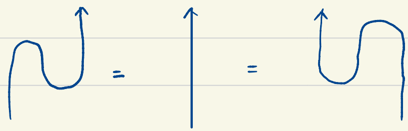
(T_5^u)



2). Two oriented sliced tangle diagrams are equivalent iff they are related by a finite sequence of Turaev moves,

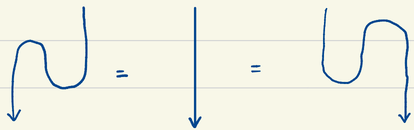
(T_0) the same as (T_0^u).

(T_1)



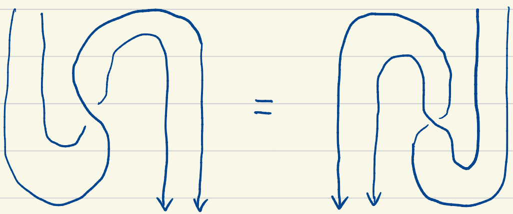
$$(\vec{n}\otimes id_+)\circ(id_+\otimes \vec{u}) = id_+ = (id_+\otimes \overleftarrow{n})\circ(\overleftarrow{u}\otimes id_+)$$

(T_2)

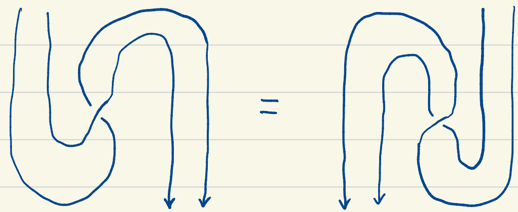


$$(\overleftarrow{n}\otimes id_-)\circ(id_-\otimes \overleftarrow{u}) = id_- = (id_-\otimes \vec{n})\circ(\vec{u}\otimes id_-)$$

(T_3)

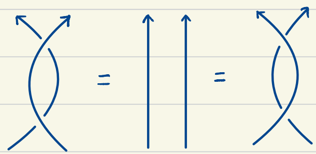


$$(id_-^{\otimes 2}\otimes \vec{n})\circ(id_-^{\otimes 2}\otimes id_+\otimes \vec{n}\otimes id_-)\circ(id_-^{\otimes 2}\otimes R^{-}\otimes id_-^{\otimes 2})$$
$$\circ\,(id_-\otimes \vec{u}\otimes id_+\otimes id_-^{\otimes 2})\circ(\vec{u}\otimes id_-^{\otimes 2})$$
$$= (\overleftarrow{n}\otimes id_-^{\otimes 2})\circ(id_-\otimes \overleftarrow{n}\otimes id_+\otimes id_-^{\otimes 2})\circ(id_-^{\otimes 2}\otimes R^{-}\otimes id_-^{\otimes 2})$$
$$\circ\,(id_-^{\otimes 2}\otimes id_+\otimes \overleftarrow{u}\otimes id_-^{\otimes 2})\circ(id_-^{\otimes 2}\otimes \overleftarrow{u})$$



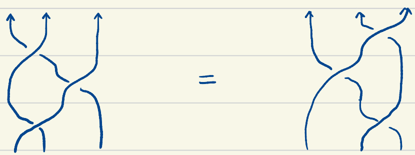
$$(id_-^{\otimes 2}\otimes \vec{n})\circ(id_-^{\otimes 2}\otimes id_+\otimes \vec{n}\otimes id_-)\circ(id_-^{\otimes 2}\otimes R\otimes id_-^{\otimes 2})$$
$$\circ\,(id_-\otimes \vec{u}\otimes id_+\otimes id_-^{\otimes 2})\circ(\vec{u}\otimes id_-^{\otimes 2})$$
$$= (\overleftarrow{n}\otimes id_-^{\otimes 2})\circ(id_-\otimes \overleftarrow{n}\otimes id_+\otimes id_-^{\otimes 2})\circ(id_-^{\otimes 2}\otimes R\otimes id_-^{\otimes 2})$$
$$\circ\,(id_-^{\otimes 2}\otimes id_+\otimes \overleftarrow{u}\otimes id_-^{\otimes 2})\circ(id_-^{\otimes 2}\otimes \overleftarrow{u})$$

(T_4)



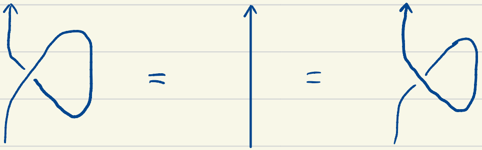
$$R\circ R^{-1} = id_+^{\otimes 2} = id_{\{+,+\}} = R^{-1}\circ R$$

(T_5)



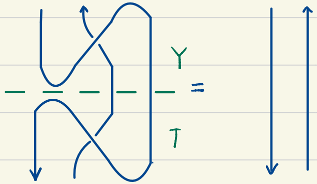
$$(R\otimes id_+)\circ(id_+\otimes R)\circ(R\otimes id_+) = (id_+\otimes R)\circ(R\otimes id_+)\circ(id_+\otimes R).$$

(T_6)

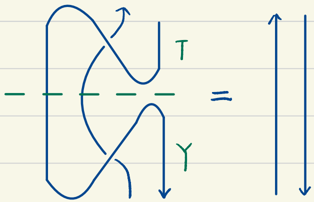


$$(id_+ \otimes \vec{n}) \circ (R \otimes id_-) \circ (id_+ \otimes \overleftarrow{u}) = id_+$$
$$= (id_+ \otimes \vec{n}) \circ (R^- \otimes id_-) \circ (id_+ \otimes \overleftarrow{u})$$

(T_7)



$$Y \circ T = id_- \otimes id_+ = id_{\{-,+\}}$$



$$T \circ Y = id_+ \otimes id_- = id_{\{+,-\}}.$$

$$\begin{cases} Y := (id_- \otimes id_+ \otimes \vec{n}) \circ (id_- \otimes R \otimes id_-) \circ (\vec{u} \otimes id_+ \otimes id_-) \\ T := (\overleftarrow{n} \otimes id_+ \otimes id_-) \circ (id_- \otimes R^- \otimes id_-) \circ (id_- \otimes id_+ \otimes \overleftarrow{u}) \end{cases}$$

Consider $(\mathbb{K}\text{-mod}^{f.f.}, \otimes_{\mathbb{K}}, I = \mathbb{K})$, a rigid symmetric monoidal category :

- Obj. : free symmetric bimodules of $\mathbb{K}$ with finite rank
- Mor. : $Hom_{\mathbb{K}}(V, W)$
- $\otimes$: $\otimes_{\mathbb{K}}$, with identity object $I = \mathbb{K}$.
- $\tau_{V,W}$: $V \otimes W \to W \otimes V$, an involution
- left/right dual : ${}^{\vee}V = V^{\vee} = V^* := Hom_{\mathbb{K}}(V, \mathbb{K})$.

<u>Def 7.</u> (operator invariant).

Fix two $\mathbb{K}$-mods V and $W \in \mathbb{K}\text{-mod}^{f.f.}$, an operator invariant is a monoidal functor $Q : \mathcal{T} \to \mathbb{K}\text{-mod}^{f.f.}$ such that,

- $Q(\emptyset) = \mathbb{K}$,
- $Q(+) = V$,
- $Q(-) = W$.

<u>Lem 8.</u> Q is determined by its images on elementary tangles and satisfies several relations induced by Turaev moves $(T_1 - T_7)$.

<u>Rem</u>: T_0 is trivial, because it is included in the defination of a monoidal functor.

- Generators:

$Q(id_+) = id_V$, $Q(id_-) = id_W$,

By abusing of notations, the images of elementary tangles are

$\vec{u} \in Hom_{\mathbb{K}}(\mathbb{K}, W \otimes V) \cong W \otimes V$

$\vec{n} \in Hom_{\mathbb{K}}(V \otimes W, \mathbb{K})$

$\overleftarrow{u} \in Hom_{\mathbb{K}}(\mathbb{K}, V \otimes W) \cong V \otimes W$

$\overleftarrow{n} \in Hom_{\mathbb{K}}(W \otimes V, \mathbb{K})$

$R \in Hom_{\mathbb{K}}(V \otimes V, V \otimes V)$, $R^{-1} \in Hom_{\mathbb{K}}(V \otimes V, V \otimes V)$.

- Relations:

(T_1) $(\vec{n} \otimes id_V) \circ (id_V \otimes \vec{u}) = id_V = (id_V \otimes \overleftarrow{n}) \circ (\overleftarrow{u} \otimes id_V)$

(T_2) $(\overleftarrow{n} \otimes id_W) \circ (id_W \otimes \overleftarrow{u}) = id_W = (id_W \otimes \vec{n}) \circ (\vec{u} \otimes id_W)$

(T_3)

$$(id_W^{\otimes 2} \otimes \vec{n}) \circ (id_W^{\otimes 2} \otimes id_V \otimes \vec{n} \otimes id_W) \circ (id_W^{\otimes 2} \otimes R^{\pm} \otimes id_W^{\otimes 2})$$
$$\circ (id_W \otimes \vec{u} \otimes id_V \otimes id_W^{\otimes 2}) \circ (\vec{u} \otimes id_W^{\otimes 2})$$
$$= (\overleftarrow{n} \otimes id_W^{\otimes 2}) \circ (id_W \otimes \overleftarrow{n} \otimes id_V \otimes id_W^{\otimes 2}) \circ (id_W^{\otimes 2} \otimes R^{\pm} \otimes id_W^{\otimes 2})$$
$$\circ (id_W^{\otimes 2} \otimes id_V \otimes \overleftarrow{u} \otimes id_W^{\otimes 2}) \circ (id_W^{\otimes 2} \otimes \overleftarrow{u})$$

(T_4) $R \circ R^{-1} = id_{V \otimes V} = R^{-1} \circ R$

(T_5) $(R \otimes id_V) \circ (id_V \otimes R) \circ (R \otimes id_V) = (id_V \otimes R) \circ (R \otimes id_V) \circ (id_V \otimes R)$.

(T_6) $(id_V \otimes \vec{n}) \circ (R \otimes id_W) \circ (id_V \otimes \overleftarrow{u}) = id_V$

$= (id_V \otimes \vec{n}) \circ (R^{-} \otimes id_W) \circ (id_V \otimes \overleftarrow{u})$

(T_7) $\begin{cases} Y := (id_W \otimes id_V \otimes \vec{n}) \circ (id_W \otimes R \otimes id_W) \circ (\vec{u} \otimes id_V \otimes id_W) \\ T := (\overleftarrow{n} \otimes id_V \otimes id_W) \circ (id_W \otimes R^{-} \otimes id_W) \circ (id_W \otimes id_V \otimes \overleftarrow{u}) \end{cases}$

$Y \circ T = id_W \otimes id_V$

$T \circ Y = id_V \otimes id_W$.

We may write $Q = Q_{V, W, \vec{n}, \vec{u}, \overleftarrow{n}, \overleftarrow{u}, R^{\pm}}$.

<u>Question</u>. How to simplify it?

<u>Idea</u>. Using rigidity!

Notation conventions. $\{v_i\}$ basis for V and $\{v^i\}$ dual basis for V^*.

$\{w_j\}$ basis for W and $\{w^j\}$ dual basis for W^*.

we shall use Einstein notation:

$\vec{u} = \vec{u}^{ij}\, w_i \otimes v_j$ $\quad$ $\overleftarrow{u} = \overleftarrow{u}^{ij}\, v_i \otimes w_j$ $\quad$ $id_V = \delta_i^j\, v_j \otimes v^i$

$\vec{n} = \vec{n}_{ij}\, v^i \otimes w^j$ $\quad$ $\overleftarrow{n} = \overleftarrow{n}_{ij}\, w^i \otimes v^j$ $\quad$ $id_W = \delta_i^j\, w_j \otimes w^i$

$R = R_{ij}^{kl}\, v^i \otimes v_k \otimes v_l \otimes v^j$ $\quad$ $R^{-1} = (R^{-1})_{ij}^{kl}\, v^i \otimes v_k \otimes v_l \otimes v^j$

Step I. $T_1, T_2 \iff \begin{cases} \delta_i^j = \vec{n}_{kl}\, \delta_m^j\, \delta_i^k\, \vec{u}^{lm} = \delta_k^j\, \overleftarrow{n}_{lm}\, \overleftarrow{u}^{kl}\, \delta_i^m \\ \delta_i^j = \delta_k^j\, \vec{n}_{lm}\, \vec{u}^{kl}\, \delta_i^m = \overleftarrow{n}_{kl}\, \delta_m^j\, \delta_i^k\, \overleftarrow{u}^{lm} \end{cases}$

$\iff \begin{cases} \vec{n}_{il}\, \vec{u}^{lj} = \delta_i^j = \overleftarrow{u}^{jl}\, \overleftarrow{n}_{li} \\ \vec{u}^{jl}\, \vec{n}_{li} = \delta_i^j = \vec{n}_{il}\, \vec{u}^{lj} \end{cases}$

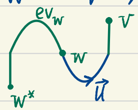
Explanation:

- Monoidal functor will preserve left/right dual, evaluation and coevaluation,
- Different left (or right) duals are isomorphic. For example,

$$\begin{tikzcd} {}^\vee A \arrow[r, "{}_A coev' \otimes id_{{}^\vee A}"] \arrow[rr, bend right, "\cong"'] & {}^\vee A' \otimes A \otimes {}^\vee A \arrow[r, "id_{{}^\vee A'} \otimes {}_A ev"] & {}^\vee A' \end{tikzcd}$$

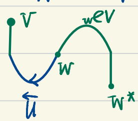
Hence, $\vec{u} = Q(coev\_)$ induces an isomorphism

$\alpha : W^* \xrightarrow{\cong} V, \quad w^i \mapsto \vec{u}^{ij} v_j$

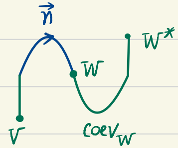


$\overleftarrow{u} = Q(\_coev)$ induces another isomorphism

$\beta : W^* \xrightarrow{\cong} V, \quad w^i \mapsto \overleftarrow{u}^{ji} v_j$

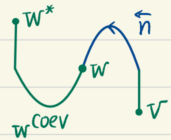


$\vec{n} = Q(ev\_)$ induce an isomorphism by



$\alpha^{-1} : v_i \mapsto \vec{n}_{ij} w^j, \quad V \to W^*$

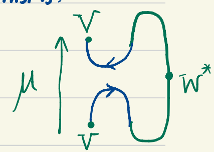
$\overleftarrow{n} = Q(\_ev)$ induce an isomorphism by



$\beta^{-1} : v_i \mapsto \overleftarrow{n}_{ji} w^j, \quad V \to W^*$.

T_1, T_2 says nothing, but two pairs of isomorphisms.

$$W^\vee \cong V \underset{\alpha}{\overset{\alpha^{-1}}{\rightleftarrows}} W^* \underset{\beta^{-1}}{\overset{\beta}{\rightleftarrows}} V \cong {}^\vee W.$$



What's more, $\vec{u}, \vec{n}, \overleftarrow{u}, \overleftarrow{n}$, can be determined by α, β.

Define

$\mu := \beta \circ \alpha^{-1} : V \to V, \quad v_i \longmapsto \underbrace{\overleftarrow{u}^{jk}\, \vec{n}_{ik}}_{\mu_i^j} v_j$

is the 'difference' between α and β.

Step II.

$$T_3 \Longleftrightarrow \overrightarrow{n}_{kj}\overrightarrow{n}_{li}(R^{\pm})^{kl}_{mn}\overrightarrow{u}^{pn}\overrightarrow{u}^{qm}\overrightarrow{n}_{\hat{n}p}\overrightarrow{n}_{\hat{m}q}\overleftarrow{u}^{\hat{r}i}\overleftarrow{u}^{\hat{s}j}$$

$$= \overleftarrow{n}_{ir}\overleftarrow{n}_{js}(R^{\pm})^{sr}_{tu}\overleftarrow{u}^{up}\overleftarrow{u}^{tq}\overrightarrow{n}_{\hat{n}p}\overrightarrow{n}_{\hat{m}q}\overleftarrow{u}^{\hat{r}i}\overleftarrow{u}^{\hat{s}j}$$

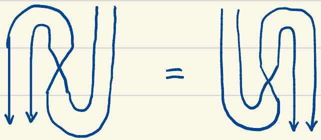
$$\Longleftrightarrow (R^{\pm})^{kl}_{mn}\overleftarrow{u}^{ri}\overrightarrow{n}_{li}\overleftarrow{u}^{sj}\overrightarrow{n}_{kj}$$

$$= (R^{\pm})^{sr}_{tu}\overleftarrow{u}^{up}\overrightarrow{n}_{np}\overleftarrow{u}^{tq}\overrightarrow{n}_{mq}$$

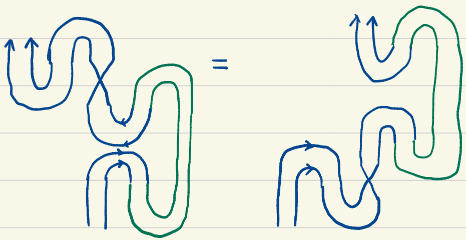
$$\Longleftrightarrow (R^{\pm})^{kl}_{mn}\mu^{r}_{l}\mu^{s}_{k} = (R^{\pm})^{sr}_{tu}\mu^{t}_{m}\mu^{u}_{n}$$

i.e., $$(\mu\otimes\mu)\circ R^{\pm} = R^{\pm}\circ(\mu\otimes\mu). \qquad (1)$$

Explanation:



$\Longleftrightarrow$



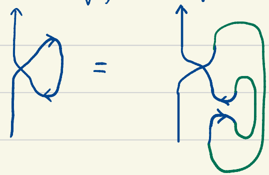
Step III

$$T_6 \Longleftrightarrow \overrightarrow{n}_{kl}(R^{\pm})^{ik}_{jm}\overleftarrow{u}^{ml} = \delta^{i}_{j}$$

$$\Longleftrightarrow (R^{\pm})^{ik}_{jm}\mu^{m}_{k} = \delta^{i}_{j}$$

i.e. $$\mathrm{Tr}_2\left(R^{\pm}\circ(id_V\otimes\mu)\right) = id_V \qquad (2)$$

Explanation:



Step IV

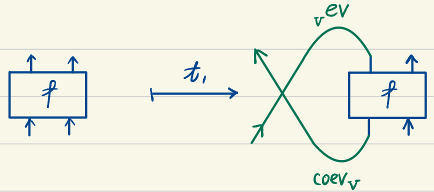
$$T_7 \Longleftrightarrow \begin{cases} \overrightarrow{n}_{lk}\overrightarrow{u}^{im}R^{pl}_{mn}(R^{-1})^{rn}_{qt}\overleftarrow{n}_{jr}\overleftarrow{u}^{tk} = \delta^{i}_{j}\delta^{p}_{q} \\ \overleftarrow{n}_{nk}\overleftarrow{u}^{mj}(R^{-1})^{ki}_{lm}R^{lr}_{pq}\overrightarrow{n}_{rs}\overrightarrow{u}^{np} = \delta^{i}_{q}\delta^{j}_{s} \end{cases}$$

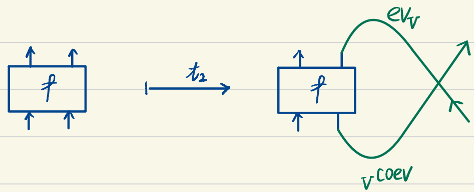
$$\Longleftrightarrow \begin{cases} \mu^{t}_{l}(\mu^{-1})^{m}_{x}R^{pl}_{mn}(R^{-1})^{yn}_{qt} = \delta^{i}_{j}\delta^{p}_{q}\overleftarrow{n}_{ix}\overleftarrow{u}^{yj} = \delta^{y}_{x}\delta^{p}_{q} \\ \mu^{m}_{x}(\mu^{-1})^{p}_{k}R^{ly}_{pq}(R^{-1})^{ki}_{lm} = \delta^{i}_{q}\delta^{j}_{s}\overrightarrow{u}^{sy}\overrightarrow{n}_{xj} = \delta^{i}_{q}\delta^{y}_{x} \end{cases}$$

In order to give a index-free formula, we introduce the following maps:

$$t_1 : \mathrm{Hom}_{\mathbb{K}}(V^{\otimes 2}, V^{\otimes 2}) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(V^{*}\otimes V, V^{*}\otimes V)$$

$$t_2 : \mathrm{Hom}_{\mathbb{K}}(V^{\otimes 2}, V^{\otimes 2}) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(V\otimes V^{*}, V\otimes V^{*})$$





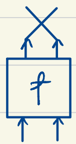
Or say, if $f : v_i \otimes v_j \mapsto f_{ij}^{kl}\, v_k \otimes v_l$, then

$f^{t_1} : v^i \otimes v_j \mapsto f_{kj}^{il}\, v^k \otimes v_l$

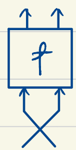
$f^{t_2} : v_i \otimes v^j \mapsto f_{il}^{kj}\, v_k \otimes v^l$

Recall $\tau : V \otimes W \to W \otimes V$

$v_i \otimes w_j \mapsto w_j \otimes v_i$.



$\tau \circ f$

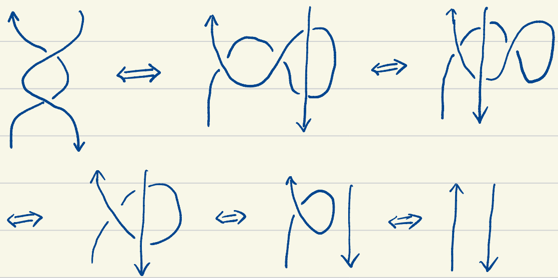


$f \circ \tau$

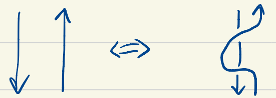
$$\begin{cases} \mu_l^t (\mu^{-1})_x^m R_{mn}^{pl} (R^{-1})_{qt}^{yn} = \delta_x^y \delta_q^p \\ \mu_x^m (\mu^{-1})_k^p R_{pq}^{ly} (R^{-1})_{lm}^{ki} = \delta_q^i \delta_x^y \end{cases}$$

$$\Longleftrightarrow \begin{cases} (\tau \circ R^{-1})^{t_1} \circ (id_{V^*} \otimes \mu) \circ (R \circ \tau)^{t_1} \circ (id_{V^*} \otimes \mu^{-1}) = id_{V^*} \otimes id_V \quad (3) \\ (R \circ \tau)^{t_1} \circ (id_{V^*} \otimes \mu^{-1}) \circ (\tau \circ R^{-1})^{t_1} \circ (id_{V^*} \otimes \mu) = id_{V^*} \otimes id_V \quad (3') \end{cases}$$

Rem. $(3) \Longleftrightarrow (3')$ is not trivial. in fact we have,



given



Thm 9. $Q_{V,W,\vec{n},\vec{u},\overleftarrow{n},\overleftarrow{u},R^{\pm}}$ is an operator invariant if and only if:

i). R is an R-matrix.

ii). there exists isomorphisms : $\alpha, \beta : W^* \to V$, s.t.

$\alpha(w^i) = \vec{u}^{ij} v_j$, $\beta(w^i) = \overleftarrow{u}^{ji} v_j$, $\alpha^{-1}(v_i) = \vec{n}_{ij} w^j$, $\beta^{-1}(v_i) = \overleftarrow{n}_{ji} w^j$.

iii). $\mu := \beta \circ \alpha^{-1} : V \to V$, satisfies :

a). $R^{\pm} \circ (\mu \otimes \mu) = (\mu \otimes \mu) \circ R^{\pm}$

b). $Tr_2 (R^{\pm} \circ (id_V \otimes \mu)) = id_V$

c). $(\tau \circ R^{-1})^{t_1} \circ (id_{V^*} \otimes \mu) \circ (R \circ \tau)^{t_1} \circ (id_{V^*} \otimes \mu^{-1}) = id_{V^*} \otimes id_V$.

Such an operator invariant will also be denoted by

$Q_{R,\alpha,\beta}$.

Coro 10. If $W = V^*$ and $Q_{R,\alpha,\beta}$ is an operator invariant, then $P_{R,\beta\circ\alpha^{-1}}$ is a link invariant and $\forall$ link L,

$$\underbrace{Q_{R,\alpha,\beta}(L)}_{\in \, End_{\mathbb{K}}(\mathbb{K})} \underset{\substack{\uparrow \\ \text{identity element in } \mathbb{K}}}{(1)} = P_{R,\mu}(L). \in \mathbb{K}.$$

Example. $\mathbb{K} = \mathbb{C}[t^{1/2}, t^{-1/2}]$, $V = \mathbb{K}e_0 \oplus \mathbb{K}e_1$, take

$$R = \begin{pmatrix} -t^{1/2} & 0 & 0 & 0 \\ 0 & -t^{1/2}+t^{3/2} & -t & 0 \\ 0 & -t & 0 & 0 \\ 0 & 0 & 0 & -t^{1/2} \end{pmatrix}$$

$$\alpha = \begin{pmatrix} -t^{-1/2} & 0 \\ 0 & -t^{1/2} \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mu = \begin{pmatrix} -t^{1/2} & 0 \\ 0 & -t^{-1/2} \end{pmatrix}$$

Then, $Q_{R,\alpha,\beta}$ is an operator invariant and

$$P_{R,\mu}(L) = (-t^{1/2} - t^{-1/2}) J(L).$$

In summary, we obtain an invariant of, not only links, but also tangles, although we need an additional relation (3). We need a better category to replace $\mathbb{K}$-mod$^{f.f.}$ to find better invariants. 'quantum invariant'.