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Drinfeld Kohno's Theorem

Recall: $\mathfrak{g}$: cpx Lie alg $\quad t$: $\mathfrak{g}$-inv symm 2-tensor

$\rightsquigarrow$ (KZ_n)

$\rightsquigarrow$ monodromy rep of $B_n = \pi_1(\mathrm{Uconf}_n(\mathbb{C})) \xrightarrow{\rho_n^{KZ}} \mathrm{Aut}_{\mathbb{C}[[h]]}(V^{\otimes n}[[h]])$

- $R_{KZ} = e^{ht/2}$
- Φ_{KZ}

$\rightsquigarrow$ $A_{\mathfrak{g},t} = (U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi_{KZ}, R_{KZ})$ q-tri q-bialg.

Today: • Representation of B_n from the universal R-matrix R_h of $U_h(\mathfrak{g})$

$\rho_n^h : B_n \to \mathrm{Aut}_{\mathbb{C}[[h]]}(V^{\otimes n}[[h]])$

$(U_h(\mathfrak{g}), \Delta_h, \varepsilon_h, 1\tilde{\otimes}1\tilde{\otimes}1, R_h)$ q-tri q-bialg.

• Limit to $\mathfrak{g}$ semisimple

$\langle , \rangle$: killing form on $\mathfrak{g}$ $\quad I_\mu$: orthonormal basis w.r.t. $\langle , \rangle$

$t = \Sigma_\mu I_\mu \otimes I_\mu$

$C = \Sigma_\mu I_\mu I_\mu \in U(\mathfrak{g})$ central elt $\quad t = \frac{1}{2}(\Delta(C) - C\otimes 1 - 1\otimes C)$

Thm (Drinfeld - Kohno)

For $\forall n > 1$, ρ_n^{KZ} and $\rho_n^{R_h}$ are equivalent for $\forall$ $\mathfrak{g}$-mod V.

i.e. $u \in \mathrm{Aut}_{\mathbb{C}[[h]]} V^{\otimes n}[[h]]$ st

$$\rho_n^{KZ}(g) = u\,\rho_n^{R_h}(g)\,u^{-1} \quad \text{for } \forall g \in B_n.$$

(Φ_{KZ}, R_{KZ}) $\qquad$ $(\Phi_h = 1\tilde{\otimes}1\tilde{\otimes}1, R_h)$

"some equivalence" between $A_{\mathfrak{g},t}$ and $U_h(\mathfrak{g})$.

Thm: $\exists$ a gauge transformation $F \in A_{\mathfrak{g},t} \tilde{\otimes} A_{\mathfrak{g},t}$ and a $\mathbb{C}[[h]]$-linear from $\alpha : U_h(\mathfrak{g}) \to (A_{\mathfrak{g},t})_F$ of quasi-triangular quasi-bialg.

§1. Braid group Representation

• A asso. alg, Δ coprod

$\rightsquigarrow$ possibility to def the tensor prod of representations

$A \curvearrowright V_1, V_2 \qquad A \xrightarrow{\Delta} A\otimes A \curvearrowright V_1 \otimes V_2$

Δ coasso. $\qquad$ tensor prod is asso.

$(\Phi = 1\otimes 1\otimes 1)$ $\qquad (V_1\otimes V_2)\otimes V_3 = V_1 \otimes (V_2 \otimes V_3)$

$\qquad\qquad a_{1,2,3}$ $\simeq$

$(\Phi$ nontrivial$)$.

$(R \cdots)$. $\qquad V\otimes W \xrightarrow{\sim} W\otimes V$

want to nontrival

braiding $\leftarrow$ $c = P_{V,W} \circ R_{V,W}$

$B_n(S)$

$(A, \Delta, \varepsilon, \Phi, R)$ $\qquad$ (Get a quasi-~~tensor~~ category)

$\exists$ a kind of gauge transf $\qquad$ The twist doesn't change the

∃ a kind of gauge transf "twist" | The twist doesn't change the relevant quasi-tensor cat.

If V is an obj of tensor cat, then $S_n \curvearrowright V^{\otimes n}$

If in a quasi-tensor cat. $B_n \curvearrowright V^{\otimes n}$?

$B_n(S) > \pi_1(UConf_n(S))$

$\langle \sigma_1 \dots \sigma_{n-1} \rangle$ $\mathbb{R}^2$

- strictness $V^{*n} = ((\cdots(V\otimes V)\otimes V)\otimes \cdots \otimes V)\otimes V$

Define autom $c_1, \dots, c_{n-1}$ of V^{*n}

$$c_i = id_{V^{*(i-1)}} * c_{V,V} * id_{V^{*(n-i-1)}}$$

In terms of the origin $V^{\otimes n}$

Petagon, hexagon.

- $c_{V,V}(v_1 \otimes v_2) = (R(v_1 \otimes v_2))_{21}$
- $c_1(v_1 \otimes \cdots \otimes v_n) = (R_{12}(v_1 \otimes \cdots \otimes v_n))_{21}$
- $i>1$ $c_i(v_1 \otimes \cdots \otimes v_n) = \Phi_i^{-1}((R_{i,i+1}\Phi_i)(v_1 \otimes \cdots \otimes v_n))_{i+1,i}$

$A^{\otimes n}$

$\Phi_i = \Delta^{(i+1)}(\Phi) \otimes 1^{\otimes(n-i-1)}$

$\Delta^{(i+1)}, A^{\otimes 3} \to A^{\otimes(i+1)}$

$\Delta^{(3)} = id_{A^{\otimes 3}}$

$\Delta^{(i+1)} = (\Delta \otimes id_A^{\otimes(i-1)})\Delta^{(i)}$

$P_n^R: B_n \to Aut(V^{\otimes n})$ $\sigma_i \mapsto c_i$

§2. Gauge Transformation

Def / Prop: $A = (A, \Delta, \varepsilon, \Phi, R)$

Let F be an invertible elt, $F \in A\otimes A$ such that

$(\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1$ (Call F a gauge transf.)

Def $\Delta_F(a) = F\cdot\Delta(a)\cdot F^{-1}$ $\quad \Delta_F: A \to A\otimes A$

$\Phi_F = F_{23}(id\otimes\Delta)(F)\,\Phi\,(\Delta\otimes id)(F^{-1})F_{12}^{-1} \in A\otimes A\otimes A$

$R_F = F_{21} R F^{-1}$

Then $A_F = (A, \Delta_F, \varepsilon, \Phi_F, R_F)$ is a q-tri q-bialg. quasi-Hopf alg.

Rmk. ① If A happens to be a bialg (i.e. $\Phi = 1$)
then A_F is not in general a bialg.

② When F is a gauge transf, so is F^{-1}.

$(A_F)_{F^{-1}} = A = (A_{F^{-1}})_F$

F' is another gauge transf, so is FF'

$(A_{F'})_F = A_{FF'}$.

Def. Two q-tri q-bi $(A, \Delta, \varepsilon, \Phi, R)$ and $(A', \Delta', \varepsilon', \Phi', R')$ are equivalent if ∃ a gauge transf F on A' and an isom $\alpha: A \to A'_F$.

- Let V be a A'-mod. By α, it becomes an A-mod. $P_n^A, P_n^{A'}: B_n \to Aut(V^{\otimes n})$
- $F_{21}R'F^{-1} = (\alpha\otimes\alpha)R$.

Thm: Let A, A' be two equiv. q-t. q-b. Then we have

$$P_n^{A'}(g)(w) = F_{12}^{-1} P_n^A(g)(F_{12}w) \quad \forall g \in B_n,\ w \in V^{\otimes n}.$$

§3. Equivalence of $(U_h(\mathfrak{g}), \Delta_h, \varepsilon_h, \Phi_h = 1\tilde{\otimes}1\tilde{\otimes}1, R_h)$

and $A_{\mathfrak{g},t} = (U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi_{KZ}, R_{KZ})$

Fact: ∃ $\mathbb{C}[[h]]$-linear isom of alg

$\alpha: U_h(\mathfrak{g}) \to U(\mathfrak{g})[[h]]$ w/ $\alpha \equiv id \mod h$

Fact : $\exists$ $\mathbb{C}[[h]]$ - linear isom of alg
$\uparrow$ $\quad \alpha : U_h(\mathfrak{g}) \longrightarrow U(\mathfrak{g})[[h]]$ w/ $\alpha \equiv id \mod h$.
($\mathfrak{g}$ semisimple) $\quad (H^i(\mathfrak{g}, U(\mathfrak{g})) = 0 \quad i=1,2.)$

Use α to transfer all str map of $U_h(\mathfrak{g})$ to $U(\mathfrak{g})[[h]]$

Rmk. $C = \Sigma I_\mu I_\mu$ central elt of $U(\mathfrak{g}) \subseteq U(\mathfrak{g})[[h]]$

$C_h = \alpha^{-1}(C)$ quantum Casimir elt of $U_h(\mathfrak{g})$

Fact : $(R_h)_{21} R_h = \Delta_h(e^{hC_h/2})(e^{-hC_h/2} \otimes e^{-hC_h/2})$

$(U_h(\mathfrak{g}), \Delta_h, \varepsilon_h, \Phi_h, R_h) \xrightarrow{\alpha} (U(\mathfrak{g})[[h]], \Delta_h^\alpha, \varepsilon_h^\alpha, 1, (\alpha\otimes\alpha)(R_h))$

where $\Delta_h^\alpha = (\alpha\otimes\alpha)\Delta_h \alpha^{-1}$

$\varepsilon_h^\alpha = \varepsilon_h \alpha^{-1}$.

3 steps to construct F

- to $(U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi, R)$ for some Φ, R. $\quad (F_1)$
- to $(U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi', R_{KZ})$ for some Φ' $\quad (F_2)$
- to $(U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi_{KZ}, R_{KZ})$ (from uniqueness thm for quantum enveloping alg) $\quad (F_3)$

F''

Set $F = F_1 F_2^{-1} F_3$

Step 1 : Δ_h, Δ_h^α are alg mor both congruent to Δ mod h.

$(\Delta(x) = x\otimes 1 + 1\otimes x)$

Prop. If α, α' are two alg mor from $U(\mathfrak{g})[[h]]$ to $U(\mathfrak{g}')[[h]]$ and $\alpha \equiv \alpha' \mod h$

If $H^1(\mathfrak{g}, U(\mathfrak{g}')) = 0$, then there exists invertible elt $G \in U(\mathfrak{g}')[[h]]$ w/ $G \equiv 1 \mod h$

such that $\alpha'(x) = G\,\alpha(x)\,G^{-1} \quad \forall x \in U(\mathfrak{g})[[h]]$

Fact : $U(\mathfrak{g}\oplus\mathfrak{g}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{g})$

So apply the proposition to $\mathfrak{g}' = \mathfrak{g}\oplus\mathfrak{g}$.

then we can find $F_1 \in U(\mathfrak{g}')[[h]] \cong (U(\mathfrak{g})\otimes U(\mathfrak{g}))[[h]]$

w/ $F_1 \equiv 1\otimes 1 \mod h$

such that $\Delta_h^\alpha(x) = F_1^{-1}\Delta_h(x) F_1$

Prop : $\varepsilon_h^\alpha := \varepsilon_h\alpha^{-1} = \varepsilon$.

Pf : Since ε_h is a counit for Δ_h, it follows that ε_h^α is a counit for Δ_h^α

$id = (\varepsilon_h^\alpha \otimes id)\Delta_h^\alpha = (\varepsilon_h^\alpha \otimes id)(F_1^{-1}\Delta F_1)$

$\forall x$

$(\varepsilon_h^\alpha \otimes id)\Delta(x) = l x l^{-1} \qquad l = (\varepsilon_h^\alpha \otimes id)(F_1)$

$$\varepsilon_h^\alpha(x) = \varepsilon_h^\alpha(\Sigma_{(x)} x'\varepsilon(x'')) = \varepsilon(\Sigma_{(x)} \varepsilon_h^\alpha(x')x'')$$
$$= \varepsilon(lxl^{-1}) = \varepsilon(l)\varepsilon(x)\varepsilon(l)^{-1} = \varepsilon(x).$$

\#

$(U_h(\mathfrak{g}), \Delta_h, \varepsilon_h, 1\hat{\otimes}1\hat{\otimes}1, R_h) \overset{\alpha}{\cong} (U(\mathfrak{g})[[h]], F_1^{-1}\Delta F_1, \varepsilon, 1\hat{\otimes}1\hat{\otimes}1, (\alpha\otimes\alpha)(R_h))$

$\downarrow$ twist by F_1

Step 1

$(U_h(\mathfrak{g}), \Delta_h, \varepsilon_h, 1\otimes 1\otimes 1, R_h) = (U(\mathfrak{g})[[h]], \alpha^{-1}\Delta\alpha, \varepsilon, 1\otimes 1\otimes 1, (\alpha\otimes\alpha)(R_h))$

$\big\downarrow$ twist by F_1

step 1

$$(U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi, R)$$

$$R = (F_1)_{21}\,(\alpha\otimes\alpha)\,(R_h)\,(F_1)^{-1}$$

Rmk. since Δ is coasso, Φ has to be $\mathfrak{g}$-inv, i.e.

$[(\Delta\otimes id)\Delta(x), \Phi] = 0$ for all $x\in\mathfrak{g}$

R is $\mathfrak{g}$-inv since Δ is co-comm. $[\Delta(x), R] = 0$.

Step 2 : (Symmetrization) Recall that $R_{KZ} = (R_{KZ})_{21}$

Prop. $\exists$ a gauge transf $F_2 \in (U(\mathfrak{g})\otimes U(\mathfrak{g}))[[h]]$ such that

$[\Delta(x), F''] = 0$ for all $x\in\mathfrak{g}$

and if we set $R' = (F_2)_{21}\,R\,F_2^{-1}$ then $R'_{21} = R'$.

$F_2 = [R(R_{21}R)^{-1/2}]^{1/2}$

Lem. $R' = R_{KZ}$.

Pf. $R'^2 = R'_{21}R' = F_2F_1(\alpha\otimes\alpha)((R_h)_{21})(F_1)_{21}^{-1}\underline{(F_2)_{21}^{-1}(F_2)_{21}}(\alpha\otimes\alpha)(R_h)F_1^{-1}F_2^{-1}$

$= F_2F_1(\alpha\otimes\alpha)\,\underline{\underline{((R_h)_{21}R_h)}}\,F_1^{-1}F_2^{-1}$

$= F_2F_1(\alpha\otimes\alpha)\,(\Delta_h(e^{hC_h/2})(e^{-hC_h/2}\otimes e^{-hC_h/2}))\,F_1^{-1}F_2^{-1}$

$= F_2F_1\,\underline{\underline{\Delta_h^{\alpha}}}(e^{h\underset{C}{\underline{\underline{\alpha(C_h)}}}/2})(e^{-h\alpha(C_h)/2}\otimes e^{-h\alpha(C_h)/2})F_1^{-1}F_2^{-1}$

$= F_2\cancel{F_1F_1^{-1}}\Delta(e^{hC/2})F_1(e^{-hC/2}\otimes e^{-hC/2})F_1^{-1}F_2^{-1}$

$= F_2\,\Delta(e^{hC/2})F_2^{-1}(e^{-hC/2}\otimes e^{-hC/2})$

$= \Delta(e^{hC/2})(e^{-hC/2}\otimes e^{-hC/2})$

$= e^{\hbar(\Delta(C)-1\otimes C-C\otimes 1)/2} = e^{ht}$

since $R'\equiv 1\otimes 1 \mod h$, it follows that $R' = e^{ht/2} = R_{KZ}$. #